Logic or Logics?

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Motivation

Classical logic

Universal logic?

(New) problems/solutions

Non-classical logics
Non-classical logics

Is there a definition?
Non-classical logics

Is there a definition?

Non-exhaustive sources

- Reject of the classic logic principles
- Reduction of the classic logical constants
- Expansion of the classical logical constants
- Reject of the classical properties of the consequence relation
- Modifications to the mathematical structure of the classical consequence relation
Non-classical logics (cont.)

Notation

<table>
<thead>
<tr>
<th>For</th>
<th>Set of well-formed formulae</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha, \beta, \delta, \ldots$</td>
<td>Formulae</td>
</tr>
<tr>
<td>$\Delta, \Gamma, \ldots$</td>
<td>Theories</td>
</tr>
<tr>
<td>$\neg, \land, \lor, \rightarrow, \bot$</td>
<td>Logical constants</td>
</tr>
<tr>
<td>$\vdash, \models, \Vdash$</td>
<td>Consequence relations</td>
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</table>
Reject of the principle of bivalence

Principle of bivalence
Every proposition is either true or false.

Many-valued logics
The number of truth values is not restricted to only two.*

- Truth values†
  - designed
  - anti-designed
  - designed and anti-designed
  - neither designed nor anti-designed
  - no designed
  - no anti-designed

---

Reject of the principle of bivalence (cont.)

- Semantical universe
  - 0: Minimal element, anti-designed and no designed
  - 1: Maximal element, designed and no anti-designed
  - \( \leq \): partial order
  - \( \forall \alpha \ (0 \leq \alpha \leq 1) \), where \( \alpha \) is the truth-value of \( \alpha \)
Reject of the principle of bivalence (cont.)

Example (Kleene’s $K_3$ logic*)

Semantic universe
\[
\begin{cases}
1 & \text{True (designed)} \\
\frac{1}{2} & \text{Undefined (anti-designed)} \\
0 & \text{False (anti-designed)}
\end{cases}
\]

<table>
<thead>
<tr>
<th></th>
<th>$\neg$</th>
<th>$\land$</th>
<th>1</th>
<th>$\frac{1}{2}$</th>
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<tr>
<td>0</td>
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A feature
There is not $\alpha$ such that $\models_{K_3} \alpha$.

Reject of the principle of explosion

Principle of explosion (pseudo-Scotus, *ex contradictione sequitur quod libet*)

\[
\forall \Gamma \ \forall \alpha \ \forall \beta \ (\Gamma, \alpha, \neg \alpha \vdash_{CL} \beta).
\]

Paraconsistent logics*†

\[
\exists \Gamma \ \exists \alpha \ \exists \beta \ (\Gamma, \alpha, \neg \alpha \nvdash_{P} \beta).
\]

Reject of the principle of explosion (cont.)

Example (da Costa’s $C_1$ logic)

Bivalent semantic for $C_1$:

A valuation for $C_1$ is a function $v : \text{For}(C_1) \to \{0, 1\}$ such that:

- $v(\alpha * \beta)$ has classical behavior ($\ast \in \{\land, \lor, \to\}$)
- $v(\alpha) = 0 \Rightarrow v(\neg \alpha) = 1, \quad v(\neg \neg \alpha) = 1 \Rightarrow v(\alpha) = 1$

A consequence

The semantic for $C_1$ is not truth-functionality:

$$v(\alpha) = 1 \not\Rightarrow v(\neg \alpha) = 1,$$
$$v(\alpha) = 1 \not\Rightarrow v(\neg \alpha) = 0.$$

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Example (da Costa’s $C'_1$ logic (cont.))

A feature

The logic $C'_1$ admits a strong negation

$$\sim \alpha \overset{\text{def}}{=} \neg \alpha \land \alpha^\circ,$$

where $\circ$ is the well-behavior operator. The negation $\sim$ is a classical negation.
Reject of the principle of the excluded third

Principle of the excluded third

$$\vdash_{CL} \alpha \lor \neg \alpha, \text{ for all formula } \alpha.$$  

Intuitionistic logics*†

- Computational meaning of the logical constants
- Proofs are constructions (programs)

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Reject of the principle of the excluded third (cont.)

The Brouwer-Heyting-Kolmogorov (BHK) interpretation

<table>
<thead>
<tr>
<th>A construction of</th>
<th>Consists of</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1 \land \alpha_2$</td>
<td>A construction of $\alpha_1$ and a construction of $\alpha_2$</td>
</tr>
<tr>
<td>$\alpha_1 \lor \alpha_2$</td>
<td>An indicator $i \in {1, 2}$ and a construction of $\alpha_i$</td>
</tr>
<tr>
<td>$\alpha_1 \rightarrow \alpha_2$</td>
<td>A method (function) which takes any construction of $\alpha_1$ to a construction of $\alpha_2$</td>
</tr>
<tr>
<td>$\bot$</td>
<td>There is not construction</td>
</tr>
<tr>
<td>$\neg \alpha$</td>
<td>A method (function) which takes any construction of $\alpha$ into a nonexistent object</td>
</tr>
<tr>
<td>$\exists x \in U. \phi(x)$</td>
<td>An element $a \in U$ and a construction of $\phi(a)$</td>
</tr>
<tr>
<td>$\forall x \in U. \phi(x)$</td>
<td>A method (function) which takes any element $x \in U$ to a construction of $\phi(x)$</td>
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</tbody>
</table>
Reject of the principle of the excluded third (cont.)

Proofs by contradiction (or *reductio ad absurdum*) and proofs of negations

Proof by contradiction

\[
\begin{align*}
\lnot \beta \\
\vdots \\
\bot \\
\hline 
\beta
\end{align*}
\]

Proof of negation

\[
\begin{align*}
\beta \\
\vdots \\
\bot \\
\hline 
\lnot \beta
\end{align*}
\]

Reject of the principle of the excluded third (cont.)

Proofs by contradiction (or *reductio ad absurdum*) and proofs of negations*

Proof by contradiction

\[
\begin{align*}
[\neg \beta] \\
\vdots \\
\bot \\
\hline \\
\beta
\end{align*}
\]

Proof of negation

\[
\begin{align*}
[\beta] \\
\vdots \\
\bot \\
\hline \\
\neg \beta
\end{align*}
\]

Justifications

\[
\begin{align*}
[\neg \beta] \\
\vdots \\
\bot \\
\hline \\
\neg \beta \rightarrow \bot \\
\hline \\
\neg \neg \beta \\
\hline \\
\beta \\
\hline \\
\beta \rightarrow \bot \\
(\bot \rightarrow \neg \alpha \rightarrow \alpha)
\end{align*}
\]

\[
\begin{align*}
[\beta] \\
\vdots \\
\bot \\
\hline \\
\beta \rightarrow \bot \\
\hline \\
\neg \beta \\
\hline \\
\neg \alpha \rightarrow \alpha
\end{align*}
\]

Reject of the principle of the excluded third (cont.)

Some features

- Since $\alpha \lor \neg \alpha$ and $\neg \neg \alpha \rightarrow \alpha$ are equivalents the proofs by contradiction are not accepted in intuitionistic logics.
Reject of the principle of the excluded third (cont.)

Some features

- Since $\alpha \lor \neg \alpha$ and $\neg \neg \alpha \rightarrow \alpha$ are equivalents the proofs by contradiction are not accepted in intuitionistic logics.
- The proofs of negations are intuitionistically valid.
Expansion of the logical constants

- Modal logics*
  \(\Box: \text{necessity}\), \(\Diamond: \text{possibility}\)
  - Temporal logics
  - Epistemic logics
  - Deontic logics

Reduction of the logical constants

Possible reductions

- Positive logics
- Implicative logics
- ...

General question

What is a logical constant? (for example \( \{\neg, \land, \lor, \rightarrow\} \))

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A general definition of logic?

Definition

A logic $\mathcal{L}$ is a structure $\mathcal{L} = \langle \text{For}, \vdash >$ where the consequence relation $\vdash \subseteq P(\text{For}) \times \text{For}$ satisfies:*†

- Reflexivity: If $\alpha \in \Gamma$, then $\Gamma \vdash \alpha$
- Monotony: If $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \alpha$
- Transitivity: If $\Gamma \vdash \alpha$ and $\Delta, \alpha \vdash \beta$, then $\Gamma, \Delta \vdash \beta$

* Béziau, Jean-Yves (2000). What is Paraconsistent Logic?
A general definition of logic?

Definition

A logic $\mathfrak{L}$ is a structure $\mathfrak{L} = \langle \text{For}, \vdash \rangle$ where the consequence relation $\vdash \subseteq P(\text{For}) \times \text{For}$ satisfies:*†

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- Transitivity: If $\Gamma \vdash \alpha$ and $\Delta, \alpha \vdash \beta$, then $\Gamma, \Delta \vdash \beta$

Remark

A ‘Tarskian logic’ is a logic whose consequence relation satisfies the above three properties.‡

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*Béziau, Jean-Yves (2000). What is Paraconsistent Logic?
Reject of the properties of the consequence relation

- Non-reflexivity logics
  - Alfabar logics*

Example

Let $\mathcal{L} = \langle \text{For}, \models \rangle$ be a logic such that $\Gamma \models \alpha$ iff exists $\Gamma'$ such that $\Gamma' \subseteq \Gamma$ and $\Gamma'$ is consistent, and $\Gamma' \vdash_{CL} \alpha$. Therefore, $p \land \neg p \not\models p \land \neg p$.

- Non-monotonic logics

  "family of formal frameworks...in which reasoners draw conclusions tentatively, reserving the right to retract them in the light of further information." †

- Non-transitive logics?

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Modifications to the mathematical structure of the consequence relation

Multiple consequence
\[ \vdash \subseteq P(\text{For}) \times P(\text{For}) \]

Sub-structural logics*

- Multi-set \( \neq \) set: \( \{A, A, B\} \neq \{A, B\} \), therefore \( \alpha, \alpha, \beta \vdash \gamma \) does not imply \( \alpha, \beta \vdash \gamma \)
- \( \alpha, \beta \vdash \gamma \) does not imply \( \beta, \alpha \vdash \gamma \)
- In general, a theory \( \Gamma \) has not to be a set

---

Towards an universal logic?

Béziau’s approach

What is...? *

Some questions

- Others approach to the consequence relations (e.g. visual inference)
- Equivalence criteria between semantics, syntax and algebra for a logic

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†Béziau, Jean-Yves (2000). What is Paraconsistent Logic?
‡Béziau, Jean-Yves (2002). Are paraconsistents negations negations?
Towards an universal logic? (cont.)

Some questions (cont.)

- Equivalence criteria between logics (e.g. Possible-translation semantics)
- Minimal properties of the logical connectives. (e.g. What is a negation?)
- Compatibility between the logical connectives
- High-order logic extensions
Possible applications

- Mathematical theories construction*
- Hypercomputation†‡

"Or maybe paraconsistent logic will save us from the tricephalous CGC-monster (CGC for Cantor-Gödel-Church) by providing foundations for finite decidable complete mathematics." §

- Type : Type¶‖

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Conclusions

- Tolerance principle in Mathematics (Newton da Costa, 1958):
  “Desde el punto de vista sintáctico-semántico, toda teoría es admisible, desde que no sea trivial. En sentido amplio, existe, en matemática, lo que no sea trivial.” *

- Logical pluralism†

- A new crisis? New opportunities?

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†Bueno, Otávio (2002). Can a Paraconsistent Theorist be a Logical Monist?
The category of the logics (bonus slides)

Definition
A category $C$ is given the following data:

- A class of objects $\text{Obj}(C)$
- A class of arrows or morphisms $\text{Mor}(C)$
- The functions $\text{dom}, \text{cod} : \text{Mor}(C) \to \text{Obj}(C)$
  Notation: $f : A \to B \equiv f \in \text{Mor}(C), \text{dom } f = A, \text{cod } f = B$
- For $A \in \text{Obj}(C)$, the identity arrow $1_A : A \to A$
- A composition operator $\circ : \text{Mor}(C) \times \text{Mor}(C) \to \text{Mor}(C)$
The category of the logics (cont.)

These data are subject to the conditions:

- \( g \circ f \) is defined iff \( \text{cod } g = \text{dom } f \)
- If \( g \circ f \) is defined, then
  \[ \text{dom}(g \circ f) = \text{dom } f \quad \text{and} \quad \text{cod}(g \circ f) = \text{cod } g \]
- For any \( f : A \rightarrow B \),
  \[ 1_B \circ f = f \quad \text{and} \quad f \circ 1_A = f \]
- For any \( f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D \),
  \[ h \circ (g \circ f) = (h \circ g) \circ f \]
Example (The category \textbf{Set})

- \textbf{Obj(\textbf{Set})}: Sets
- \textbf{Mor(\textbf{Set})}: functions
- The identity arrow $1_A$: The identity function
- The composition operator $\circ$: The composition of functions

Technical remark

The usual definition of a function $f : A \rightarrow B$ as a set $f \subseteq A \times B$ which is \textcolor{red}{single-valued} and \textcolor{red}{totally defined} is not sufficient to uniquely determine \textcolor{red}{\text{cod} \ f}$. Therefore it is necessary to define $f$ as a triple $(A, \text{graph}(f), B)$. 
The category of the logics (cont.)

Example (The category 3)
The category of the logics (cont.)

Example

Almost every known example of a mathematical structure with the appropriate structure-preserving map yields a category.

<table>
<thead>
<tr>
<th>Category</th>
<th>Objects</th>
<th>Morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Set</strong></td>
<td>Sets</td>
<td>Functions</td>
</tr>
<tr>
<td><strong>Pfn</strong></td>
<td>Sets</td>
<td>Partial functions</td>
</tr>
<tr>
<td><strong>Vect</strong></td>
<td>Vector spaces</td>
<td>Linear transforms</td>
</tr>
<tr>
<td><strong>Top</strong></td>
<td>Topological spaces</td>
<td>Continuous functions</td>
</tr>
<tr>
<td><strong>Poset</strong></td>
<td>Posets</td>
<td>Monotone functions</td>
</tr>
<tr>
<td><strong>CPO</strong></td>
<td>Complete posets</td>
<td>Continuous functions</td>
</tr>
<tr>
<td><strong>Lat</strong></td>
<td>Lattices</td>
<td>Structure preserving homomorphisms</td>
</tr>
</tbody>
</table>
The category of the logics (cont.)

Example

A deductive system $\vdash_D$ can be turned on a category $D$

- $\text{Obj}(D)$: Formulae
- $\text{Mor}(D)$: Proofs
- The identity arrow $1_A : A \to A$: A proof of $A \vdash_D A$
- The composition operator $\circ$: Transitivity of the $\vdash_D$

\[
\begin{align*}
  f : A &\to B \\
  g : B &\to C
\end{align*}
\]

\[
  g \circ f : A \to C
\]
References


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