Logic or Logics?

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Introduction

Classical Logic \rightarrow Universal Logic \rightarrow Non-Classical Logics

New problems/New solutions
Some sources for non-classical logics

- Reject of the classic logic principles.
- Reduction of the classic logical constants.
- Expansion of the classical logical constants.
- Reject of the classical properties of the consequence relation.
- Modifications to the mathematical structure of the classical consequence relation.
Non-Classical Logics

Graham Priest (1948 - )
Non-Classical Logics

Notation

Set of well-formed formulae $\mathcal{F}$
Formulae $\alpha, \beta, \delta, \ldots$
Theories $\Delta, \Gamma, \ldots$
Logical constants $\neg, \wedge, \vee, \rightarrow, \bot$
Consequence relations $\vdash, \models, \nvDash$
Reject of the Principle of Bivalence

Principle of bivalence

Every proposition is either true or false.
Reject of the Principle of Bivalence

Many-valued logics
The number of truth values is not restricted to only two. See, e.g. (Rescher 1969; Peña 1993).

- Truth values (Peña 1993, pp. 33-35)
  - designed
  - anti-designed
  - designed and anti-designed
  - neither designed nor anti-designed
  - no designed
  - no anti-designed
Many-valued logics (continuation)

Semantical universe (Peña 1993, p. 21)

(i) 0: Minimal element, anti-designed and no designed.
(ii) 1: Maximal element, designed and no anti-designed.
(iii) $\forall \alpha (0 \leq |\alpha| \leq 1)$, where $|\alpha|$ is the truth-value of $\alpha$ and $\leq$ is a partial or total order.
Reject of the Principle of Bivalence

Example (Kleene’s $K_3$ logic)

- Semantical universe

<table>
<thead>
<tr>
<th></th>
<th>true</th>
<th>designed</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>½</td>
<td>undefined</td>
<td>anti-designed</td>
</tr>
<tr>
<td>0</td>
<td>false</td>
<td>anti-designed</td>
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</tbody>
</table>
Reject of the Principle of Bivalence

Example (Kleene’s $K_3$ logic)

Semantical universe

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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>undefined</td>
<td>anti-designed</td>
</tr>
<tr>
<td>0</td>
<td>false</td>
<td>anti-designed</td>
</tr>
</tbody>
</table>

Truth tables

<table>
<thead>
<tr>
<th></th>
<th>$\neg$</th>
<th>$\wedge$</th>
<th>$\vee$</th>
<th>$\rightarrow$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
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<td>$\frac{1}{2}$</td>
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<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Example (continuation)

- A feature
  There is not $\alpha$ such that $\models_{K_3} \alpha$.
- See, e.g. (Epstein 1990).
Reject of the Principle of Explosion

Principle of explosion (pseudo-Scotus, *ex contradictione sequitur quod libet*)

\[ \forall \Gamma \forall \alpha \forall \beta (\Gamma, \alpha, \neg \alpha \vdash_{CL} \beta). \]

Paraconsistent logics

\[ \exists \Gamma \exists \alpha \exists \beta (\Gamma, \alpha, \neg \alpha \not\vdash_{P} \beta). \]

See, e.g. (Bobenrieth 1996) and (Carnielli and Marcos 2002).
Reject of the Principle of Explosion

Example (da Costa’s $C_1$ logic)

- Bivalent semantic for $C_1$

  A valuation for $C_1$ is a function

  \[ v : \mathcal{F}(C_1) \to \{0, 1\} \]

  such that:

  (i) $v(\alpha \ast \beta)$ has classical behavior ($\ast \in \{\land, \lor, \to\}$)

  (ii) for negation

  \[
  v(\alpha) = 0 \Rightarrow v(\neg \alpha) = 1, \\
  v(\neg \neg \alpha) = 1 \Rightarrow v(\alpha) = 1.
  \]
Example (continuation)

▶ A consequence
The semantic for $C_1$ is not truth-functionality:

\[ v(\alpha) = 1 \not\Rightarrow v(\neg\alpha) = 1, \]
\[ v(\alpha) = 1 \not\Rightarrow v(\neg\alpha) = 0. \]

▶ A feature
The logic $C_1$ admits a strong negation

\[ \sim \alpha \overset{\text{def}}{=} \neg\alpha \land \alpha^\circ, \]

where $^\circ$ is the well-behavior operator. The negation $\sim$ is a classical negation.

▶ See, e.g. (Marcos 1999, p. 47).
Reject of the Principle of the Excluded Third

Principle of the excluded third

$$\vdash_{\text{CL}} \alpha \lor \neg \alpha, \text{ for all formula } \alpha.$$  

Intuitionistic logics

- Computational meaning of the logical constants
- Propositions-as-types correspondence

See, e.g. (van Dalen 2013) and (Sørensen and Urzyczyn 2006).
Reject of the Principle of the Excluded Third

The Brouwer-Heyting-Kolmogorov (BHK) interpretation

<table>
<thead>
<tr>
<th>A construction of</th>
<th>Consists of</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_1 \land \alpha_2 )</td>
<td>A construction of ( \alpha_1 ) and a construction of ( \alpha_2 ).</td>
</tr>
<tr>
<td>( \alpha_1 \lor \alpha_2 )</td>
<td>An indicator ( i \in {1, 2} ) and a construction of ( \alpha_i ).</td>
</tr>
<tr>
<td>( \alpha_1 \rightarrow \alpha_2 )</td>
<td>A method (function) which takes any construction of ( \alpha_1 ) to a construction of ( \alpha_2 ).</td>
</tr>
<tr>
<td>\bot )</td>
<td>There is not construction.</td>
</tr>
<tr>
<td>( \neg \alpha \stackrel{\text{def}}{=} \alpha \rightarrow \bot )</td>
<td>A method (function) which takes any construction of ( \alpha ) into a nonexistent object.</td>
</tr>
</tbody>
</table>
### Reject of the Principle of the Excluded Third

The Brouwer-Heyting-Kolmogorov (BHK) interpretation (continuation)

<table>
<thead>
<tr>
<th>A construction of</th>
<th>Consists of</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\exists x \in U. \varphi(x)$</td>
<td>An element $a \in U$ and a construction of $\varphi(a)$.</td>
</tr>
<tr>
<td>$\forall x \in U. \varphi(x)$</td>
<td>A method (function) which takes any element $x \in U$ to a construction of $\varphi(x)$.</td>
</tr>
</tbody>
</table>
Reject of the Principle of the Excluded Third

Proofs by contradiction (or *reductio ad absurdum*) and proofs of negations

**Proof by contradiction**

\[
\begin{array}{c}
[\neg \beta] \\
\vdots \\
\bot \\
\hline \\
\beta \\
\end{array}
\]

**Proof of negation (Bauer 2017)**

\[
\begin{array}{c}
[\beta] \\
\vdots \\
\bot \\
\hline \\
\neg \beta \\
\end{array}
\]
Reject of the Principle of the Excluded Third

Justifications

Proof by contradiction

\[
\begin{align*}
[\neg \beta] \\
\vdots \\
\bot \\
\hline
\neg \beta \rightarrow \bot \\
\hline
\neg \neg \beta \\
\hline
\beta (\vdash \neg \neg \alpha \rightarrow \alpha)
\end{align*}
\]

Proof of negation

\[
\begin{align*}
[\beta] \\
\vdots \\
\bot \\
\hline
\beta \rightarrow \bot \\
\hline
\neg \beta (\neg \alpha \overset{\text{def}}{=} \alpha \rightarrow \bot)
\end{align*}
\]
Reject of the Principle of the Excluded Third

Some features

- Since $\alpha \lor \neg\alpha$ and $\neg\neg\alpha \rightarrow \alpha$ are equivalents the proofs by contradiction are not accepted in intuitionistic logics.
Reject of the Principle of the Excluded Third

Some features

▶ Since $\alpha \lor \neg \alpha$ and $\neg \neg \neg \alpha \rightarrow \alpha$ are equivalents the proofs by contradiction are not accepted in intuitionistic logics.

▶ The proofs of negations are intuitionistically valid.
Expansion of the Logical Constants

Example

Modal logics (Hughes and Cressivell 1998)

(□: necessity, ◊: possibility)

▶ Temporal logics
▶ Epistemic logics
▶ Deontic logics
Reduction of the Logical Constants

Possible reductions

- Positive logics
- Implicative logics
- ...

See, e.g. (Rasiowa 1974).

General question
What is a logical constant? (for example \{\neg, \land, \lor, \rightarrow\})
A General Definition of Logic?

Definition

A logic $\mathcal{L}$ is a structure $\mathcal{L} = \langle F, \vdash \rangle$ where the consequence relation $\vdash$ defined on $\mathcal{P}(F) \times F$ satisfies (Carnielli and Marcos 2002; Gabbay 1994; Béziau 2000):

- If $\alpha \in \Gamma$, then $\Gamma \vdash \alpha$ (reflexivity)
- If $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \alpha$ (monotony)
- If $\Gamma \vdash \alpha$ and $\Delta, \alpha \vdash \beta$, then $\Gamma, \Delta \vdash \beta$ (transitivity)

Remark

A ‘Tarksian logic’ is a logic whose consequence relation satisfies the above three properties (Carnielli and Matulovic 2015). See also (Béziau 2005).
A General Definition of Logic?

Definition
A logic \( \mathcal{L} \) is a structure \( \mathcal{L} = \langle \mathcal{F}, \vdash \rangle \) where the consequence relation \( \vdash \) defined on \( P(\mathcal{F}) \times \mathcal{F} \) satisfies (Carnielli and Marcos 2002; Gabbay 1994; Béziau 2000):

- If \( \alpha \in \Gamma \), then \( \Gamma \vdash \alpha \) (reflexivity)
- If \( \Gamma \vdash \alpha \) and \( \Gamma \subseteq \Delta \), then \( \Delta \vdash \alpha \) (monotony)
- If \( \Gamma \vdash \alpha \) and \( \Delta, \alpha \vdash \beta \), then \( \Gamma, \Delta \vdash \beta \) (transitivity)

Remark
A ‘Tarskian logic’ is a logic whose consequence relation satisfies the above three properties (Carnielli and Matulovic 2015). See also (Béziau 2005).
Reject of Properties of the Consequence Relation

Non-reflexivity logics

Example
Let $\mathcal{L} = \langle F, \vdash \rangle$ be a logic such that $\Gamma \vdash \alpha$ iff exists $\Gamma'$ such that
(i) $\Gamma' \subseteq \Gamma$,
(ii) $\Gamma'$ is consistent and
(iii) $\Gamma' \vdash_{CL} \alpha$.
Therefore, $p \land \neg p \not\vdash p \land \neg p$ (Krause and Béziau 1997).
Reject of Properties of the Consequence Relation

Non-monotonic logics

‘family of formal frameworks... in which reasoners draw conclusions tentatively, reserving the right to retract them in the light of further information.’ (Strasser and Antonelli 2014)
Non-transitive logics

Weber (2017) mentions some non-transitive logics by Smiley (1959) and Ripley. See also (Weir 2015) and (Ripley 2018).
Modifications to the Mathematical Structure of the Consequence Relation

Multiple consequence
$\models \subseteq P(\mathcal{F}) \times P(\mathcal{F})$

Sub-structural logics

- Multi-set $\neq$ set: $(\{A, A, B\} \neq \{A, B\})$, therefore $\alpha, \alpha, \beta \models \gamma$ does not imply $\alpha, \beta \models \gamma$.
- $\alpha, \beta \models \gamma$ does not imply $\beta, \alpha \models \gamma$.
- In general, a theory $\Gamma$ has not to be a set.

See, e.g. (Restall 2004).
Béziau’s ‘approach’

(i) Béziau (2000). ‘What is Paraconsistent Logic?’
(ii) Béziau, de Freitas and Viana (2001). ‘What is Classical Propositional Logic? (A Study in Universal Logic)’.
(iii) Béziau (2002). ‘Are Paraconsistents Negations Negations?’
Some questions

(i) Other approaches to the consequence relations (e.g. visual inference).
(ii) Equivalence criteria between semantics, syntax and algebra for a logic.
(iii) Equivalence criteria between logics (e.g. possible-translation semantics).
(iv) Minimal properties of the logical connectives (e.g. what is a negation?).
(v) Compatibility between the logical connectives.
(vi) High-order logic extensions.
Universal logic is not itself a system of logic; it is a general study of the various systems of logic, considered as logical structures, in the same way that universal algebra is a general study of algebras considered as algebraic structures. Universal logic promotes unity in diversity not by reducing everything to one system but by developing concepts in a general framework to have a better understanding of the universe of logic systems. (Béziau 2023, p. 150)
Possible Applications

- Mathematical theories construction (Mortensen 1995).
- Hypercomputation (Sylvan and Copeland 1998; Agudelo and Sicard 2004)
- ‘Or maybe paraconsistent logic will save us from the tricephalous CGC-monster (CGC for Cantor-Gödel-Church) by providing foundations for finite decidable complete mathematics.’ (Béziau 1999, p. 16)
Conclusions

▶ Tolerance principle in Mathematics (Newton da Costa, 1958):

‘Desde el punto de vista sintáctico-semántico, toda teoría es admisible, desde que no sea trivial. En sentido amplio, existe, en matemática, lo que no sea trivial.’ (Bobenrieth 1996, p. 180)

▶ Logical pluralism. See, e.g. (Bueno 2002).

▶ A new crisis? New opportunities?
A Category of Logics (Bonus Slides)

Definition
A category $C$ is given by the following data:

- A class of objects $\text{Obj}(C)$.
- A class of arrows or morphisms $\text{Mor}(C)$.
- The functions $\text{dom}, \text{cod} : \text{Mor}(C) \to \text{Obj}(C)$.

Notation:

$$f : A \to B \equiv f \in \text{Mor}(C), \quad \text{dom } f = A, \quad \text{cod } f = B.$$ 

- For $A \in \text{Obj}(C)$, the identity arrow $\text{id}_A : A \to A$.
- A composition operator $\circ : \text{Mor}(C) \times \text{Mor}(C) \to \text{Mor}(C)$. 
A Category of Logics

These data are subject to the following conditions:

- \( g \circ f \) is defined iff \( \text{cod} \ g = \text{dom} \ f \).
- If \( g \circ f \) is defined, then
  \[
  \text{dom}(g \circ f) = \text{dom} \ f \quad \text{and} \quad \text{cod}(g \circ f) = \text{cod} \ g.
  \]
- For any \( f : A \to B \),
  \[
  \text{id}_B \circ f = f \quad \text{and} \quad f \circ \text{id}_A = f.
  \]
- For any \( f : A \to B, g : B \to C, h : C \to D \),
  \[
  h \circ (g \circ f) = (h \circ g) \circ f.
  \]
Example (The category **Set**)

- **Obj**(Set): Sets
- **Mor**(Set): functions
- The identity arrow \( \text{id}_A \): The identity function
- The composition operator \( \circ \): The composition of functions

**Technical remark**

The usual definition of a function \( f : A \to B \) as a set \( f \subseteq A \times B \) which is **single-valued** and **totally defined** is not sufficient to uniquely determine \( \text{cod} \, f \). Therefore it is necessary to define \( f \) as a triple \((A, \text{graph}(f), B)\).
A Category of Logics

Example (The category 3)

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow{id_A} & & \uparrow{id_C} \\
B & \xrightarrow{f} & \uparrow{id_B} & \xleftarrow{g} & C \\
\end{array}
\]
A Category of Logics

Example

Almost every known example of a mathematical structure with the appropriate structure-preserving map yields a category.

<table>
<thead>
<tr>
<th>Category</th>
<th>Objects</th>
<th>Morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Set</strong></td>
<td>Sets</td>
<td>Functions</td>
</tr>
<tr>
<td><strong>Pfn</strong></td>
<td>Sets</td>
<td>Partial functions</td>
</tr>
<tr>
<td><strong>Vect</strong></td>
<td>Vector spaces</td>
<td>Linear transforms</td>
</tr>
<tr>
<td><strong>Top</strong></td>
<td>Topological spaces</td>
<td>Continuous functions</td>
</tr>
<tr>
<td><strong>Poset</strong></td>
<td>Posets</td>
<td>Monotone functions</td>
</tr>
<tr>
<td><strong>CPO</strong></td>
<td>Complete posets</td>
<td>Continuous functions</td>
</tr>
<tr>
<td><strong>Lat</strong></td>
<td>Lattices</td>
<td>Structure preserving homomorphisms</td>
</tr>
</tbody>
</table>
A Category of Logics

Example

A deductive system $\vdash_D$ can be turned on a category $\mathbf{D}$

- **Obj($\mathbf{D}$)**: Formulae
- **Mor($\mathbf{D}$)**: Proofs
- The identity arrow $\text{id}_A : A \rightarrow A$: A proof of $A \vdash_D A$
- The composition operator $\circ$: Transitivity of the $\vdash_D$

$$f : A \rightarrow B \quad g : B \rightarrow C$$

$$g \circ f : A \rightarrow C$$


References


