Logic or Logics?

Andrés Sicard-Ramírez

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Motivation

Classical logic

Universal logic?

(New) problems/solutions

Non-classical logics
Non-classical logics

Is there a definition?

Sources
(Non-exhaustive!)

- Reject of the classic logic principles
- Reduction of the classic logical constants
- Expansion of the classical logical constants
- Reject of the classical properties of the consequence relation
- Modifications to the mathematical structure of the classical consequence relation
Non-classical logics (cont.)

Notation

- For
  - $\alpha, \beta, \delta, \ldots$ Set of well-formed formulae
  - $\Delta, \Gamma, \ldots$ Formulae
  - $\neg, \wedge, \vee, \rightarrow$ Theories
  - $\vdash, \models, \vDash$ Logical connectives
  - $\vdash, \models, \vDash$ Consequence relations
Reject of the principle of bivalence

Principle of bivalence
Every proposition is either true or false.

Many-valued logics
The number of truth values is not restricted to only two.*

- Truth values†
  - designed
  - anti-designed
  - designed and anti-designed
  - neither designed nor anti-designed
  - no designed
  - no anti-designed

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Semantical universe
- 0: Minimal element, anti-designed and no designed
- 1: Maximal element, designed and no anti-designed
- ≤: partial order
- ∀α (0 ≤ /α/ ≤ 1), where /α/ is the truth-value of α
Reject of the principle of bivalence (cont.)

Example (Kleene’s $K_3$ logic*)

Semantic universe \[
\begin{align*}
&\{1\quad \text{True (designed)} \\
&\frac{1}{2}\quad \text{Undefined (anti-designed)} \\
&0\quad \text{False (anti-designed)}
\end{align*}
\]

|   | $\neg$ | $\land$ | 1 | $\frac{1}{2}$ | 0 | $\lor$ | 1 | $\frac{1}{2}$ | 0 | $\rightarrow$ | 1 | $\frac{1}{2}$ | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | $\frac{1}{2}$ | 0 | 1 | 1 | 1 | 1 | 1 | $\frac{1}{2}$ | 0 |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | $\frac{1}{2}$ | 0 | 0 | 1 | 1 | 1 |

A feature

There is not $\alpha$ such that $\models_{K_3} \alpha$.

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Reject of the principle of explosion

Principle of explosion (pseudo-Scotus, *ex contradictione sequitur quod libet*)

\[ \forall \Gamma \ \forall \alpha \ \forall \beta \ (\Gamma, \alpha, \neg \alpha \vdash_{CL} \beta). \]

Paraconsistent logics*†

\[ \exists \Gamma \ \exists \alpha \ \exists \beta \ (\Gamma, \alpha, \neg \alpha \nvdash_P \beta). \]

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*Bobenrieth, M. Andrés (1996). ¿Inconsistencias, Por Qué No?  
Reject of the principle of explosion (cont.)

Example (da Costa’s $C_1$ logic)

Bivalent semantic for $C_1$:

A valuation for $C_1$ is a function $v : \text{For}(C_1) \rightarrow \{0, 1\}$ such that:

- $v(\alpha \ast \beta)$ has classical behavior ($\ast \in \{\land, \lor, \rightarrow\}$)
- $v(\alpha) = 0 \Rightarrow v(\neg \alpha) = 1, \quad v(\neg \neg \alpha) = 1 \Rightarrow v(\alpha) = 1$

A consequence

The semantic for $C_1$ is not truth-functionality:

$$v(\alpha) = 1 \not\Rightarrow v(\neg \alpha) = 1,$$
$$v(\alpha) = 1 \not\Rightarrow v(\neg \alpha) = 0.$$
Reject of the principle of explosion (cont.)

Example (da Costa’s $C_1$ logic (cont.))

A feature

The logic $C_1$ admits a strong negation $\sim \alpha \overset{\text{def}}{=} \neg \alpha \land \alpha^\circ$, where $\circ$ is the well-behavior operator. The negation $\sim$ is a classical negation.
Reject of the principle of the excluded third

Principle of the excluded third

\[ \forall \alpha \ (\vdash_{CL} \alpha \lor \neg \alpha). \]

Intuitionistic logics*†

- Computational meaning of the logical constants
- Proofs are constructions (programs)

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Reject of the principle of the excluded third (cont.)

The Brouwer-Heyting-Kolmogorov (BHK) interpretation

<table>
<thead>
<tr>
<th>A construction of</th>
<th>Consists of</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1 \land \alpha_2$</td>
<td>A construction of $\alpha_1$ and a construction of $\alpha_2$</td>
</tr>
<tr>
<td>$\alpha_1 \lor \alpha_2$</td>
<td>An indicator $i \in {1, 2}$ and a construction of $\alpha_i$</td>
</tr>
<tr>
<td>$\alpha_1 \rightarrow \alpha_2$</td>
<td>A method (function) which takes any construction of $\alpha_1$ to a construction of $\alpha_2$</td>
</tr>
<tr>
<td>$\bot$</td>
<td>There is not construction</td>
</tr>
<tr>
<td>$\neg \alpha$</td>
<td>A method (function) which takes any construction of $\alpha$ into a nonexistent object</td>
</tr>
<tr>
<td>$\exists x \in U. \phi(x)$</td>
<td>An element $a \in U$ and a construction of $\phi(a)$</td>
</tr>
<tr>
<td>$\forall x \in U. \phi(x)$</td>
<td>A method (function) which takes any element $x \in U$ to a construction of $\phi(x)$</td>
</tr>
</tbody>
</table>
Reject of the principle of the excluded third (cont.)

A feature

The proofs by contradiction are invalids

Prove $\alpha$

$$\neg \alpha$$

$$\vdots$$

$$\bot$$

$$\neg \alpha \rightarrow \bot \equiv \neg \alpha$$

Invalid!

But, disprove $\alpha$ is valid

$$\alpha$$

$$\vdots$$

$$\bot$$

$$\alpha \rightarrow \bot \equiv \neg \alpha$$
Expansion of the logical constants

- Modal logics*
  (□ : necessity, ◇ : possibility)
  - Temporal logics
  - Epistemic logics
  - Deontic logics

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Reduction of the logical constants

Possible reductions*

- Positive logics
- Implicative logics
- ...

General question

What is a logical constant? (for example \(\{\neg, \land, \lor, \rightarrow\}\))

A general definition of logic?

Definition

A logic \( \mathcal{L} \) is a structure \( \mathcal{L} = \langle \text{For}, \vdash \rangle \) where the consequence relation \( \vdash \subseteq \mathcal{P}(\text{For}) \times \text{For} \) satisfies:*†

- Reflexivity: If \( \alpha \in \Gamma \), then \( \Gamma \vdash \alpha \)
- Monotony: If \( \Gamma \vdash \alpha \) and \( \Gamma \subseteq \Delta \), then \( \Delta \vdash \alpha \)
- Transitivity: If \( \Gamma \vdash \alpha \) and \( \Delta, \alpha \vdash \beta \), then \( \Gamma, \Delta \vdash \beta \)

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* Béziau, Jean-Yves (2000). What is Paraconsistent Logic?
A general definition of logic?

Definition

A logic $\mathcal{L}$ is a structure $\mathcal{L} = \left< \text{For}, \models \right>$ where the consequence relation $\models \subseteq P(\text{For}) \times \text{For}$ satisfies:*†

- Reflexivity: If $\alpha \in \Gamma$, then $\Gamma \models \alpha$
- Monotony: If $\Gamma \models \alpha$ and $\Gamma \subseteq \Delta$, then $\Delta \models \alpha$
- Transitivity: If $\Gamma \models \alpha$ and $\Delta, \alpha \models \beta$, then $\Gamma, \Delta \models \beta$

Remark

A ‘Tarskian logic’ is a logic whose consequence relation satisfies the above three properties.‡

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* Béziau, Jean-Yves (2000). *What is Paraconsistent Logic?*
Reject of the properties of the consequence relation

- Non-reflexivity logics
  - Alfabar logics*

Example

Let $\mathcal{L} = \langle \text{For}, \models \rangle$ be a logic such that $\Gamma \models \alpha$ iff exists $\Gamma'$ such that $\Gamma' \subseteq \Gamma$ and $\Gamma'$ is consistent, and $\Gamma' \vdash_{CL} \alpha$. Therefore, $p \land \neg p \not\models p \land \neg p$.

- Non-monotonic logics

  "family of formal frameworks...in which reasoners draw conclusions tentatively, reserving the right to retract them in the light of further information." †

- Non-transitive logics?

Modifications to the mathematical structure of the consequence relation

Multiple consequence
$\models \subseteq P(\text{For}) \times P(\text{For})$

Sub-structural logics*

- Multi-set $\neq$ set: $(\{A, A, B\} \neq \{A, B\})$, therefore $\alpha, \alpha, \beta \models \gamma$ does not imply $\alpha, \beta \models \gamma$
- $\alpha, \beta \models \gamma$ does not imply $\beta, \alpha \models \gamma$
- In general, a theory $\Gamma$ has not to be a set

Towards an universal logic?

Béziau’s approach

What is...? *†‡

Some questions

- Others approach to the consequence relations (e.g. visual inference)
- Equivalence criteria between semantics, syntax and algebra for a logic

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†Béziau, Jean-Yves (2000). What is Paraconsistent Logic?
‡Béziau, Jean-Yves (2002). Are paraconsistents negations negations?
Towards an universal logic? (cont.)

Some questions (cont.)

- Equivalence criteria between logics (e.g. Possible-translation semantics)
- Minimal properties of the logical connectives. (e.g. What is a negation?)
- Compatibility between the logical connectives
- High-order logic extensions
Possible applications

- Mathematical theories construction*
- Hypercomputation†‡

"Or maybe paraconsistent logic will save us from the tricephalous CGC-monster (CGC for Cantor-Gödel-Church) by providing foundations for finite decidable complete mathematics." §

- Type : Type¶‖

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Conclusions

- Tolerance principle in Mathematics (Newton da Costa, 1958):
  “Desde el punto de vista sintáctico-semántico, toda teoría es admisible, desde que no sea trivial. En sentido amplio, existe, en matemática, lo que no sea trivial.” *

- Logical pluralism †

- A new crisis? New opportunities?

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† Bueno, Otávio (2002). Can a Paraconsistent Theorist be a Logical Monist?
The category of the logics (bonus slides)

Definition

A category $\mathcal{C}$ is given the following data:

- A class of objects $\text{Obj}(\mathcal{C})$
- A class of arrows or morphisms $\text{Mor}(\mathcal{C})$
- The functions $\text{dom}, \text{cod} : \text{Mor}(\mathcal{C}) \to \text{Obj}(\mathcal{C})$
  
  Notation: $f : A \to B \equiv f \in \text{Mor}(\mathcal{C})$, $\text{dom} \ f = A$, $\text{cod} \ f = B$
- For $A \in \text{Obj}(\mathcal{C})$, the identity arrow $1_A : A \to A$
- A composition operator $\circ : \text{Mor}(\mathcal{C}) \times \text{Mor}(\mathcal{C}) \to \text{Mor}(\mathcal{C})$
The category of the logics (cont.)

These data are subject to the conditions:

- $g \circ f$ is defined iff $\text{cod } g = \text{dom } f$
- If $g \circ f$ is defined, then $\text{dom}(g \circ f) = \text{dom } f$ and $\text{cod}(g \circ f) = \text{cod } g$
- For any $f : A \rightarrow B$, $1_B \circ f = f$ and $f \circ 1_A = f$
- For any $f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D$, $h \circ (g \circ f) = (h \circ g) \circ f$
The category of the logics (cont.)

Example (The category \( \textbf{Set} \))

- \( \text{Obj(} \textbf{Set} \text{)} \): Sets
- \( \text{Mor(} \textbf{Set} \text{)} \): functions
- The identity arrow \( 1_A \): The identity function
- The composition operator \( \circ \): The composition of functions

Technical remark

The usual definition of a function \( f : A \to B \) as a set \( f \subseteq A \times B \) which is single-valued and totally defined is not sufficient to uniquely determine \( \text{cod } f \). Therefore it is necessary to define \( f \) as a triple \( (A, \text{graph}(f), B) \).
The category of the logics (cont.)

Example (The category 3)

\[
\begin{array}{c}
A \xrightarrow{h} C \\
B \xrightarrow{g} C \\
A \xleftarrow{f} B \\
\end{array}
\]
The category of the logics (cont.)

Example

Almost every known example of a mathematical structure with the appropriate structure-preserving map yields a category.

<table>
<thead>
<tr>
<th>Category</th>
<th>Objects</th>
<th>Morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set</td>
<td>Sets</td>
<td>Functions</td>
</tr>
<tr>
<td>Pfn</td>
<td>Sets</td>
<td>Partial functions</td>
</tr>
<tr>
<td>Vect</td>
<td>Vector spaces</td>
<td>Linear transforms</td>
</tr>
<tr>
<td>Top</td>
<td>Topological spaces</td>
<td>Continuous functions</td>
</tr>
<tr>
<td>Poset</td>
<td>Posets</td>
<td>Monotone functions</td>
</tr>
<tr>
<td>CPO</td>
<td>Complete posets</td>
<td>Continuous functions</td>
</tr>
<tr>
<td>Lat</td>
<td>Lattices</td>
<td>Structure preserving homomorphisms</td>
</tr>
</tbody>
</table>
The category of the logics (cont.)

Example

A deductive system \( \vdash_D \) can be turned on a category \( D \)

- **Obj\((D)\)**: Formulae
- **Mor\((D)\)**: Proofs
- The identity arrow \( 1_A : A \to A \): A proof of \( A \vdash_D A \)
- The composition operator \( \circ \): Transitivity of the \( \vdash_D \)

\[
\frac{f : A \to B \quad g : B \to C}{g \circ f : A \to C}
\]
References


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