Lambda Calculus and Combinatory Logic

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Introduction

References

- Textbook: Hindley, J. Roger and Seldin, Jonathan P. [2008]. Lambda-Calculus and Combinators. An Introduction. Cambridge University Press.
- Barendregt, Henk and Barendsen, Erik [2000]. Introduction to Lambda Calculus. Revisited edition.
- Barendregt, H. P. [1981] [2004]. The Lambda Calculus. Its Syntax and Semantics. Revised edition, 6th impression. Vol. 103. Studies in Logic and the Foundations of Mathematics. Elsevier.

Lambda Calculus and Combinatory Logic

- "Two systems of logic which can also serve as abstract programming languages." [Hindley and Seldin 2008, p. ix]
- The goal was to use them in the foundation of mathematics.

Lambda Calculus

Invented by Alonzo Church (around 1930s).

- The goal was to use it in the foundation of mathematics. Intended for studying functions and recursion.
- Computability model.
- Model of untyped functional programming languages.

What is the Combinatory Logic?


Invented by Moses Schönfinkel (1920) and Haskell Curry (1927).

- Intended for clarify the role of quantified variables.
- Idea: To do logic and mathematics without use bound variables.
- Combinators: Operators which manipulate expressions by cancellation, duplication, bracketing and permutation.

Lambda Calculus

Introduction

- λ-calculus is a collection of several formal systems
- λ-notation
  - Anonymous functions
  - Currying

Definition (λ-terms)

$$
\begin{aligned}
v \in V &\Rightarrow v \in \lambda\text{-terms} && \text{(atom)}\\
c \in C &\Rightarrow c \in \lambda\text{-terms} && \text{(atom)}\\
M, N \in \lambda\text{-terms} &\Rightarrow (MN) \in \lambda\text{-terms} && \text{(application)}\\
M \in \lambda\text{-terms}, x \in V &\Rightarrow (\lambda x.M) \in \lambda\text{-terms} && \text{(abstraction)}
\end{aligned}
$$

where V/C is a set of variables/constants.

Introduction

Conventions and syntactic sugar

- Application associates to the left
  $MN_1N_2 \ldots N_k$ means $(...((MN_1)N_2)...N_k)$
- Application has higher precedence
  $\lambda x.PQ$ means $(\lambda x.(PQ))$
- $\lambda x_1 x_2 \ldots x_n.M$ means $(\lambda x_1.(\lambda x_2.(\ldots(\lambda x_n.M)\ldots)))$
- $M \equiv N$ means the syntactic identity

Example

$(\lambda xyz.xz(yz))uvw \equiv ((((\lambda x.(\lambda y.(\lambda z.((xz)(yz)))))u)v)w)$.

Term-Structure and Substitution

Definition (P occurs in Q)

- P occurs in P
- If P occurs in M or in N, then P occurs in (MN)
- If P occurs in M or $P \equiv x$, then P occurs in $(\lambda x.M)$

Definition (scope)

In $\lambda x.M$, M is called the scope of λx.

Term-Structure and Substitution

Definition (free and bound occurrence of variables)

An occurrence of a variable x in a term P is called

- bound if it is in the scope of a λx in P
- bound and binding, iff it is the x in λx
- free otherwise

Definition (bound variable of P)

If x has at least one binding occurrence in P.

Definition (free variable of P)

If x has at least one free occurrence in P.

$\mathrm{FV}(P)$: The set of free variables of P.

Term-Structure and Substitution

Example

$(\lambda y.yx(\lambda x.y(\lambda y.z)x))vw$. (whiteboard)

Definition (close term or combinator)

A term without free variables.

Term-Structure and Substitution

Definition (substitution $[N/x]M$)

The result of substituting N for every free occurrence of x in M, and changing bound variables to avoid clashes.

$$
\begin{aligned}
[N/x]x &\equiv N && \\
[N/x]a &\equiv a && \text{for all atoms } a \not\equiv x \\
[N/x](PQ) &\equiv ([N/x]P\ [N/x]Q) && \\
[N/x](\lambda x.P) &\equiv (\lambda x.P) && \\
[N/x](\lambda y.P) &\equiv (\lambda y.P) && y \not\equiv x, x \notin \mathrm{FV}(P) \\
[N/x](\lambda y.P) &\equiv \lambda y.[N/x]P && y \not\equiv x, x \in \mathrm{FV}(P), y \notin \mathrm{FV}(N) \\
[N/x](\lambda y.P) &\equiv \lambda z.[N/x][z/y]P && y \not\equiv x, x \in \mathrm{FV}(P), y \in \mathrm{FV}(N)
\end{aligned}
$$

where in the last equation, z is chosen to be a variable $\notin \mathrm{FV}(NP)$.

Term-Structure and Substitution

Example

$[(\lambda y.vy)/x](y(\lambda v.xv)) \equiv y(\lambda z.(\lambda y.vy)z)$ (with $z \not\equiv v, y, x$).

Term-Structure and Substitution

Definition (α-conversion or changed of bound variables)

Replace $\lambda x.M$ by $\lambda y.[y/x]M$ ($y \notin \mathrm{FV}(M)$).

Definition (α-congruence ($P \equiv_\alpha Q$))

P is changed to Q by a finite (perhaps empty) series of α-conversions.

Beta Reduction

Definition (β-contraction ($P \triangleright_{1\beta} Q$))

Replace an occurrence of $(\lambda x.M)N$ (β-redex) in P by $[N/x]M$ (contractum).

Example

Whiteboard.

Definition (β-reduction ($P \triangleright_{\beta} Q$))

P is changed to Q by a finite (perhaps empty) series of β-contractions and α-conversions.

Example

$(\lambda x.(\lambda y.yx)z)v \triangleright_{\beta} zv$.

Beta Reduction

Definition (β-normal form)

A term which contains no β-redex.

β-nf: The set of all β-normal forms.

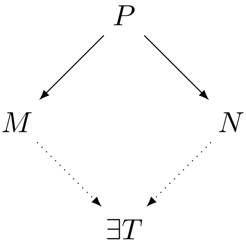
Example

Whiteboard.

Beta Reduction

Theorem (Church-Rosser theorem for $\triangleright_\beta$)

$$\frac{P \triangleright_\beta M \qquad P \triangleright_\beta N}{\exists T. M \triangleright_\beta T \wedge N \triangleright_\beta T}$$



Corollary

If P has a β-normal form, it is unique modulo $\equiv_\alpha$; that is, if P has β-normal forms M and N, then $M \equiv_\alpha N$.

Beta Equality

Definition (β-equality or β-convertibility ($P =_\beta Q$))

Exist $P_0, \ldots, P_n$ such that

- $P_0 \equiv P$
- $P_n \equiv Q$
- $(\forall i \leq n-1)(P_i \triangleright_{1\beta} P_{i+1} \quad \vee \quad P_{i+1} \triangleright_{1\beta} P_i \quad \vee \quad P_i \equiv_\alpha P_{i+1})$

Theorem (Church-Rosser theorem for $=_\beta$)

$$\frac{P =_\beta Q}{\exists T. P \triangleright_\beta T \wedge Q \triangleright_\beta T}$$

Proof

Whiteboard.

Beta Equality

Corollary

If $P, Q \in \beta$-nf and $P =_\beta Q$, then $P \equiv_\alpha Q$.

Corollary

The relation $=_\beta$ is non-trivial (not all terms are β-convertible to each other).

Proof

Whiteboard.

Combinatory Logic

Introduction

Idea

To do logic and mathematics without use bound variables.

Combinators

Operators which manipulate expressions by cancellation, duplication, bracketing and permutation.

Introduction

Example (informal)

The commutative law for addition

$$\forall xy. x + y = y + x,$$

can be written as

$$A = \mathsf{C}\, A,$$

where Axy represents $x + y$ and C is a combinator with the property

$$\mathsf{C}\, f\, x\, y = f\, y\, x.$$

Introduction

Example (some combinators (informal))

$\mathsf{B}\, f\, g\, x = f\,(g\, x)$	composition operator
$\mathsf{B}'\, f\, g\, x = g\,(f\, x)$	reversed composition operator
$\mathsf{I}\, x = x$	identity operator
$\mathsf{K}\, x\, y = x$	projection operator
$\mathsf{S}\, f\, g\, x = f\, x\,(g\, x)$	stronger composition operator
$\mathsf{W} f\, x = f\, x\, x$	doubling operator

Introduction

Definition (CL-terms)

$$v \in V \Rightarrow v \in \text{CL-terms}$$
$$c \in C \Rightarrow c \in \text{CL-terms}$$
$$X, Y \in \text{CL-terms} \Rightarrow (X\,Y) \in \text{CL-terms}$$

where

$$V: \text{ Set of variables}$$
$$C = \{\mathsf{I}, \mathsf{K}, \mathsf{S}, \dots\}: \text{ Set of atomic constants}$$

$\mathrm{FV}(X)$: The set of variables occurring in X.

Introduction

Definition (atoms, basic combinators and combinator)

An atom is a variable or atomic constant. The basic combinators are I, K and S. A combinator is a CL-term whose only atoms are basic combinators.

Introduction

Definition (substitution $[U/x]Y$)

The result of substituting U for every occurrence of x in Y:

$$
\begin{aligned}
[U/x]\, x &\equiv U \\
[U/x]\, a &\equiv a && \text{for all atoms } a \not\equiv x \\
[U/x]\,(V\, W) &\equiv ([U/x]\, V)\,([U/x]\, W)
\end{aligned}
$$

Weak Reduction

Definition (weak redex)

The CL-terms $\mathsf{I}\,X$, $\mathsf{K}\,X\,Y$ and $\mathsf{S}\,X\,Y\,Z$.

Definition (weak contraction $(U \triangleright_{1w} V)$)

Replace an occurrence of a weak redex in U using:

$$\begin{aligned} \mathsf{I}\,X &\text{ by } X, \\ \mathsf{K}\,X\,Y &\text{ by } X, \\ \mathsf{S}\,X\,Y\,Z &\text{ by } X\,Z\,(Y\,Z). \end{aligned}$$

Weak Reduction

Definition (weak reduction ($U \triangleright_w V$))

The CL-term U is changed to V by a finite (perhaps empty) series of weak contractions.

Definition (weak normal form)

A CL-term which contains no weak redex.

Weak Reduction

Example

Let $\mathsf{B} \equiv \mathsf{S}\,(\mathsf{K}\,\mathsf{S})\,\mathsf{K}$. Then $\mathsf{B}\,X\,Y\,Z \triangleright_w X\,(Y\,Z)$ (whiteboard).

Weak Reduction

Example

Let $\mathsf{B} \equiv \mathsf{S}\,(\mathsf{K}\,\mathsf{S})\,\mathsf{K}$. Then $\mathsf{B}\,X\,Y\,Z \rhd_w X\,(Y\,Z)$ (whiteboard).

Example

Let $\mathsf{W} \equiv \mathsf{S}\,\mathsf{S}\,(\mathsf{K}\,\mathsf{I})$. Then

i) $\mathsf{W}\,X\,Y \rhd_w X\,Y\,Y$ and
ii) $\mathsf{W}\,\mathsf{W}\,\mathsf{W} \rhd_w \mathsf{W}\,\mathsf{W}\,\mathsf{W} \rhd_w \cdots$

Weak Reduction

Theorem (substitution theorem for $\triangleright_w$)

$$X \triangleright_w Y \Rightarrow [U/x]\,X \triangleright_w [U/x]\,Y.$$

Theorem (Church-Rosser theorem for $\triangleright_w$)

$$\frac{P \triangleright_w M \qquad P \triangleright_w N}{\exists T.M \triangleright_w T \wedge N \triangleright_w T}$$

Corollary (uniqueness of nf)

A CL-term can have at most one weak normal form.

Abstraction

Idea

To define a term $[x].M$ such that

$$([x].M)\,N \;\triangleright_w\; [N/x]\,M.$$

Definition (abstraction)

For every term M and every variable x,

$$
\begin{aligned}
[x].M &\equiv \mathsf{K}\,M && \text{if } x \notin \mathrm{FV}(M) && (1)\\
[x].x &\equiv \mathsf{I} && && (2)\\
[x].U\,x &\equiv U && \text{if } x \notin \mathrm{FV}(U) && (3)\\
[x].U\,V &\equiv \mathsf{S}\,([x].U)\,([x].V) && \text{if neither (1) nor (3) applies} && (4)
\end{aligned}
$$

Abstraction

Example

$[x].x\,y \equiv \mathsf{S}\,\mathsf{I}\,(\mathsf{K}\,y)$ (whiteboard).

Abstraction

Theorem

For every term M and every variable x, $[x].M$ is always defined, does not contain x and $([x].M)\,x \rhd_w M$.

Proof

Whiteboard.

Theorem

For every term M and every variable x,

$$([x].M)\,N \rhd_w [N/x]\,M.$$

Notation

$[x_1, x_2, \ldots, x_n].M \equiv [x_1].([x_2].(\ldots([x_n].M)\ldots)).$

Abstraction

Example

$[x, y].x\,y\,y \equiv \mathsf{S}\,\mathsf{S}\,(\mathsf{K}\,\mathsf{I}) \equiv \mathsf{W}$ (whiteboard).

Weak Equality

Definition (weak equality or weak convertibility $(X =_w Y)$)

Exist $X_0, \ldots, X_n$ such that

i) $X_0 \equiv X$

ii) $X_n \equiv Y$

iii) $(\forall i \leq n-1)(X_i \rhd_{1w} X_{i+1} \quad \vee \quad X_{i+1} \rhd_{1w} X_i)$

Theorem (Church-Rosser theorem for $=_w$)

$$\frac{X =_w Y}{\exists T. X \rhd_w T \wedge Y \rhd_w T}$$

Corollary

If X and Y are distinct weak normal forms, them $X \neq_w Y$; in particular $\mathsf{S} \neq_w \mathsf{K}$. Hence $=_w$ is non-trivial in the sense that not all terms are weakly equal.

Weak Equality

About the 'weak' adjective

$$X =_\beta Y \Rightarrow \lambda x.X =_\beta \lambda x.Y,$$

but

$$X =_w Y \not\Rightarrow [x].X =_w [x].Y.$$

Example

Let $X \equiv \mathsf{S}\,x\,y\,z$ and $Y \equiv x\,z\,(y\,z)$. Then $X =_w Y$, but $[x].X \neq_w [x].Y$, where

$$[x].X \equiv \mathsf{S}\,(\mathsf{S}\,\mathsf{S}\,(\mathsf{K}\,y))\,(\mathsf{K}\,z),$$
$$[x].Y \equiv \mathsf{S}\,(\mathsf{S}\,\mathsf{I}\,(\mathsf{K}\,z))\,(\mathsf{K}\,(y\,z)).$$

The Power of λ

Introduction

Notation	Meaning for λ	Meaning for CL
term	λ-term	CL-term
$X \equiv Y$	$X \equiv_\alpha Y$	X is identical to Y
$X \triangleright_{\beta,w} Y$	$X \triangleright_\beta Y$	$X \triangleright_w Y$
$X =_{\beta,w} Y$	$X =_\beta Y$	$X =_w Y$
λx	λx	$[x]$

The Fixed-Point Theorem

Idea

For every term X there is a term P such

$$X\,P =_{\beta,w} P.$$

The term P is called a fixed-point of X.

The Fixed-Point Theorem

Theorem (fixed-point theorem)

There is a combinator Y such that for every term X

1. $\mathsf{Y}\,X =_{\beta,w} X\,(\mathsf{Y}\,X)$.
2. $\mathsf{Y}\,X \triangleright_{\beta,w} X\,(\mathsf{Y}\,X)$.

The Fixed-Point Theorem

Theorem (fixed-point theorem)

There is a combinator Y such that for every term X

1. $\mathsf{Y}\,X =_{\beta,w} X\,(\mathsf{Y}\,X)$.
2. $\mathsf{Y}\,X \rhd_{\beta,w} X\,(\mathsf{Y}\,X)$.

Proof.

$\mathsf{Y}_{\mathrm{Turing}} \equiv U\,U$, where $U \equiv \lambda u.\lambda x.x\,(u\,u\,x)$ (whiteboard). ∎

The Fixed-Point Theorem

Corollary

For every term Z and $n \geq 0$, the equation

$$x\, y_1 \ldots y_n = Z$$

can be solved for x. That is, there is a term X such that

$$X\, y_1 \ldots y_n =_{\beta,w} [X/x]\, Z.$$

The Fixed-Point Theorem

Corollary

For every term Z and $n \geq 0$, the equation

$$x\, y_1 \ldots y_n = Z$$

can be solved for x. That is, there is a term X such that

$$X\, y_1 \ldots y_n =_{\beta,w} [X/x]\, Z.$$

Proof.

$X \equiv \mathsf{Y}\, (\lambda.x\, y_1 \ldots\, y_n.Z)$ (whiteboard). ∎

The Fixed-Point Theorem

Definition (fixed-point combinator)

A fixed-point combinator is any combinator Y such $Y\,X =_{\beta,w} X\,(\mathsf{Y}\,X)$, for all terms X.

The Fixed-Point Theorem

Example

$\mathsf{Y}_{\text{Curry-Rosenbloom}} \equiv \lambda x.V\,V$, where $V \equiv \lambda y.x\,(y\,y)$ is a fixed-point combinator. (Whiteboard)

Böhms's Theorem

Definition (η-redex)

In λ-calculus, a λ-term of form $\lambda x.M\,x$ with $x \notin \mathrm{FV}(M)$ is called an η-redex and is said to η-contract to M.

Definition ($\beta\eta$-normal forms)

In λ-calculus, a λ-term which contains no β-redex and no η-redex.

$\beta\eta$-nf: The set of all $\beta\eta$-normal forms.

Example

The λ-term $\lambda u.\lambda x.u\,x$ is in β-nf but not in $\beta\eta$-nf.

Böhms's Theorem

Definition (strong normal forms)

In CL, the class of strong nf is defined inductively by

- All atoms other than I, K and S are in strong nf.
- If $X_1, \ldots, X_n$ are in strong nf, and a is any atom $\not\equiv$ I, K, S, then $a\, X_1 \ldots X_n$ is in strong nf.
- If X is in strong nf, then so is $[x].X$.

Böhms's Theorem

Theorem (Böhms's theorem)

Let M and N be combinators, either in $\beta\eta$-normal form (in λ) or in strong normal form (in CL). If $M \not\equiv N$, then there exists $n \geq 0$ and combinators $L_1, \ldots, L_n$ such that

$$M\, L_1 \ldots L_n\, x\, y \rhd_{\beta,w}\; x,$$
$$N\, L_1 \ldots L_n\, x\, y \rhd_{\beta,w}\; y.$$

Böhms's Theorem

Corollary

Let M and N be distinct combinators in $\beta\eta$-normal form (in λ) or in strong normal form (in CL). If we add the equation $M = N$ as a new axiom to the definition $=_\beta$ or $=_w$, then all terms become equal.

Proof

Whiteboard.

Leftmost Reduction

Idea

Proving that a given term has no normal form.

Definition (contraction $(X \triangleright_R Y)$)

$(X \triangleright_R Y)$: R is an redex in X and Y is the result of contracting R in X.

Example

$(\lambda x.(\lambda y.y\,x)\,z)\,v \triangleright_{(\lambda y.y\,x)\,z} (\lambda x.z\,x)\,v$.

Leftmost Reduction

Definition (reduction)

A reduction ρ is

$$\begin{aligned} \mathrm{CL}: &\qquad X_1 \triangleright_{R_1} X_2 \triangleright_{R_2} \cdots \\ \lambda: &\qquad X_1 \triangleright_{R_1} Y_1 \equiv_\alpha X_2 \triangleright_{R_2} \cdots \end{aligned}$$

Leftmost Reduction

Definition

Length of a reduction: The number of its contractions.

Terminus: The last term of a reduction of length finite.

A reduction ρ has maximal length iff either ρ is infinite or its terminus contains no redexes.

A redex is maximal iff it is not contained in any other redex.

A (maximal) redex is the left most maximal redex iff it is the leftmost of the maximal redexes.

Leftmost reduction: In every contraction, the contracted redex is the leftmost maximal redex.

Leftmost Reduction

Example

Let $X \equiv \mathsf{S}\,(\mathsf{I}\,(\mathsf{K}\,x\,y))\,(\mathsf{I}\,z)$.

Redexes: $\mathsf{I}\,(\mathsf{K}\,x\,y)$, $\mathsf{K}\,x\,y$ and $\mathsf{I}\,z$.
Maximal redexes: $\mathsf{I}\,(\mathsf{K}\,x\,y)$ and $\mathsf{I}\,z$.
Leftmost redex: $\mathsf{I}\,(\mathsf{K}\,x\,y)$.

Leftmost Reduction

Example

Let $X \equiv \mathsf{S}\,(\mathsf{I}\,(\mathsf{K}\,x\,y))\,(\mathsf{I}\,z)$.

Redexes: $\mathsf{I}\,(\mathsf{K}\,x\,y)$, $\mathsf{K}\,x\,y$ and $\mathsf{I}\,z$.
Maximal redexes: $\mathsf{I}\,(\mathsf{K}\,x\,y)$ and $\mathsf{I}\,z$.
Leftmost redex: $\mathsf{I}\,(\mathsf{K}\,x\,y)$.

Example

The leftmost reduction for X.

$$\begin{aligned} \mathsf{S}\,(\underline{\mathsf{I}\,(\mathsf{K}\,x\,y)})(\mathsf{I}\,z) &\rhd_{1w} \mathsf{S}\,(\underline{\mathsf{K}\,x\,y})(\mathsf{I}\,z) \\ &\rhd_{1w} \mathsf{S}\,x\,(\underline{\mathsf{I}\,z}) \\ &\rhd_{1w} \mathsf{S}\,x\,z \end{aligned}$$

Leftmost Reduction

Theorem (leftmost reduction theorem)

If a term X has a normal form X^*, then the leftmost reduction of X is finite and ends at X^*.

Representing the Computable Functions

Representability

Definition

Let X, Y be λ-terms or CL-terms. Then

$$X^0 Y \equiv Y,$$
$$X^{n+1} Y \equiv X(X^n Y).$$

Definition (Church numerals)

For λ: $\quad \overline{n} \equiv \lambda xy.x^n y,$

For CL: $\quad \overline{n} \equiv (\mathsf{SB})^n(\mathsf{KI})$, where $\mathsf{B} \equiv \mathsf{S}(\mathsf{KS})\mathsf{K}$.

Representability

Definition (representability)

Let φ be a partial function $\varphi : \mathbb{N}^m \to \mathbb{N}$. A term X represents φ iff

$$\varphi(n_1, \ldots, n_m) = p \Rightarrow X\overline{n_1} \ldots \overline{n_m} =_{\beta,w} \overline{p},$$
$$\varphi(n_1, \ldots, n_m) \text{ does not exist} \Rightarrow X\overline{n_1} \ldots \overline{n_m} \text{ has no nf.}$$

Representability

Example

The successor function $\sigma(n) = n + 1$ is represented by

$$\text{In } \lambda: \quad \overline{\sigma} \equiv \lambda uxy.x(uxy) \quad \text{(whiteboard)}$$
$$\text{In CL}: \quad \overline{\sigma} \equiv \mathsf{SB}$$

Definition (conditional operator)

$$\mathsf{D} \equiv \lambda xyz.z(\mathsf{K}y)x$$

For all X, Y

$$\mathsf{D}XY\overline{0} =_{\beta,w} X \quad \text{(whiteboard)}$$
$$\mathsf{D}XY\overline{k+1} =_{\beta,w} Y \quad \text{(whiteboard)}$$

$\mathsf{D}XY\overline{n}$ is called if $n = 0$ then X, else Y.

Recursion Using Fixed-Points

Example (informal)

(From: Peyton Jones [1987])

$$\mathsf{fac} \equiv \lambda n.\text{if } n = 0 \text{ then } 1 \text{ else } n * \mathsf{fac}\ (n-1)$$

$$\mathsf{fac} \equiv \lambda n.(\ldots \mathsf{fac} \ldots)$$

$$\mathsf{fac} \equiv (\lambda f n.(\ldots f \ldots))\mathsf{fac}$$

$$h \equiv \lambda f n.(\ldots f \ldots) \quad \text{(not recursive!)}$$

$$\mathsf{fac} \equiv h\ \mathsf{fac} \quad (\mathsf{fac} \text{ is a fixed-point of } h!)$$

$$\mathsf{fac} \equiv \mathsf{Y}h$$

Recursion Using Fixed-Points

Example (cont.)

$$
\begin{aligned}
\mathsf{fac}\ 1 &\equiv \mathsf{Y} h\ 1 \\
&=_{\beta,w} h(\mathsf{Y} h)\ 1 \\
&\equiv (\lambda f n.(\dots f \dots))(\mathsf{Y} h)\ 1 \\
&\rhd_{\beta,w} \text{if } 1 = 0 \text{ then } 1 \text{ else } 1 * (\mathsf{Y} h\ 0) \\
&\rhd_{\beta,w} 1 * (\mathsf{Y} h\ 0) \\
&=_{\beta,w} 1 * (h(\mathsf{Y} h)\ 0) \\
&\equiv 1 * ((\lambda f n.(\dots f \dots))(\mathsf{Y} h) 0) \\
&\rhd_{\beta,w} 1 * (\text{if } 0 = 0 \text{ then } 1 \text{ else } 1 * (\mathsf{Y} h\ (-1))) \\
&\rhd_{\beta,w} 1 * 1 \\
&\rhd_{\beta,w} 1
\end{aligned}
$$

Representing the Computable Functions

Theorem (representation of Turing-computable functions)

In λ or CL every Turing-computable function can be represented by a combinator.

The Formal Theories $\lambda\beta$ and CLw

The Definitions of the Theories

Definition ($\lambda\beta$, formal theory of β-equality)

Formulas: $M = N$, where $M, N \in \lambda$-terms.

Axiom-schemes:

$$
\begin{array}{lll}
(\alpha) & \lambda x.M & = \lambda y.[y/x]M \text{ if } y \in \mathrm{FV}(M), \\
(\beta) & (\lambda x.M)N & = [N/x]M, \\
(\rho) & M & = M.
\end{array}
$$

Rules of inference:

$$(\mu)\ \frac{M = M'}{NM = NM'} \qquad (\xi)\ \frac{M = M'}{\lambda x.M = \lambda x.M'} \qquad (\tau)\ \frac{M = N \qquad N = P}{M = P}$$

$$(\nu)\ \frac{M = M'}{MN = M'N} \qquad (\sigma)\ \frac{M = N}{N = M}$$

The Definitions of the Theories

Definition ($\lambda\beta$, formal theory of β-equality)

Deductions: $\lambda\beta, A_1, \ldots, A_n \vdash B$ (There is a deduction of B from the assumptions $A_1, \ldots, A_n$ in $\lambda\beta$).

Theorems: $\lambda\beta \vdash B$ (The formula B is probable in $\lambda\beta$).

The Definitions of the Theories

Example

Let M and N be two closed terms

$$\dfrac{\dfrac{(\lambda x.(\lambda y.x))M = [M/x]\lambda y.x \equiv \lambda y.M}{(\lambda x.(\lambda y.x))MN = (\lambda y.M)N}\,(\nu) \qquad (\lambda y.M)N = [N/y]M \equiv M}{(\lambda x.(\lambda y.x))MN = M}\,(\tau)$$

That is to say, $\lambda\beta \vdash (\lambda xy.x)MN = M$.

Remark

$\lambda\beta$ is a equational theory and it is a logic-free theory (there are not logical connectives or quantifiers in its formulae).

The Definitions of the Theories

Definition ($\lambda\beta$, formal theory of β-reduction)

(Similar to the formal theory of β-equality, but:

1. Formulas: $M \triangleright_\beta N$.
2. To change '$=$' by '$\triangleright_\beta$.
3. Remove the rule (σ).)

Formulas: $M \triangleright_\beta N$, where $M, N \in \lambda$-terms.

Axiom-schemes:

$$\begin{array}{lll} (\alpha) & \lambda x.M & \triangleright_\beta \lambda y.[y/x]M \text{ if } y \in \mathrm{FV}(M), \\ (\beta) & (\lambda x.M)N & \triangleright_\beta [N/x]M, \\ (\rho) & M & \triangleright_\beta M. \end{array}$$

The Definitions of the Theories

Definition ($\lambda\beta$, formal theory of β-reduction)

Rules of inference:

$$(\mu)\ \frac{M \triangleright_\beta M'}{NM \triangleright_\beta NM'} \qquad (\xi)\ \frac{M \triangleright_\beta M'}{\lambda x.M \triangleright_\beta \lambda x.M'}$$

$$(\nu)\ \frac{M \triangleright_\beta M'}{MN \triangleright_\beta M'N} \qquad (\tau)\ \frac{M \triangleright_\beta N \qquad N \triangleright_\beta P}{M \triangleright_\beta P}$$

Theorem

$$M \triangleright_\beta N \iff \lambda\beta \vdash M \triangleright_\beta N,$$
$$M =_\beta N \iff \lambda\beta \vdash M = N.$$

The Definitions of the Theories

Definition (CLw, formal theory of weak equality)

Formulas: $M = N$, where $M, N \in \mathrm{CL}$-terms

Axiom-schemes:

$$
\begin{array}{rll}
(\mathsf{I}) & \mathsf{I}X & = X, \\
(\mathsf{K}) & \mathsf{K}XY & = X, \\
(\mathsf{S}) & \mathsf{S}XYZ & = XZ(YZ), \\
(\rho) & X & = X.
\end{array}
$$

Rules of inference: The same rules than the theory $\lambda\beta$ except the rule (ξ).

Deductions: $\mathrm{CLw}, A_1, \dots, A_n \vdash B$.

Theorems: $\mathrm{CLw} \vdash B$.

The Definitions of the Theories

Definition (CLw, formal theory of weak reduction)

Similar to the formal theory of β-reduction.

Theorem

$$M \triangleright_w N \iff \mathrm{CLw} \vdash M \triangleright_w N,$$
$$M =_w N \iff \mathrm{CLw} \vdash M = N.$$

Equivalence of Theories

$\mathcal{T}$: Theory

$\mathcal{F}$: Formulas of $\mathcal{T}$

Rule of inference $\mathcal{R}(\varphi)$: Given by a partial function $\varphi : \mathcal{F}^n \to \mathcal{F}$

Instance of $\mathcal{R}(\varphi)$:

$$\frac{A_1, \ldots, A_n}{B}$$

Equivalence of Theories

Notation

$\mathcal{T}, \mathcal{T}'$: Theories with the same formulas
$\mathcal{R}$: Inference rule
C: Formula

Definition (derivable rules)

$\mathcal{R}$ is derivable in $\mathcal{T}$ iff for each instance of $\mathcal{R}$ (with premises $A_1, \ldots, A_n$ and conclusion B)

$$\mathcal{T}, A_1, \ldots, A_n \vdash B.$$

Definition (admissible rules)

$\mathcal{R}$ is admissible in $\mathcal{T}$ iff adding $\mathcal{R}$ to $\mathcal{T}$ as a new rule will not increase the set of theorems of $\mathcal{T}$.

Definition (derivable and admissible formula)

$$\mathcal{T} \vdash C.$$

Equivalence of Theories

Theorem

If $\mathcal{R}$ is derivable in $\mathcal{T}$, then $\mathcal{R}$ is admissible in $\mathcal{T}$. The implication in the opposite direction does not hold in general.

Definition (theories theorem-equivalent)

$\mathcal{T}$ and $\mathcal{T}'$ are theorem-equivalent iff every rule and axiom of $\mathcal{T}$ is admissible in $\mathcal{T}'$ and vice-versa.

Definition (theories rule-equivalent)

$\mathcal{T}$ and $\mathcal{T}'$ are rule-equivalent iff every rule and axiom of $\mathcal{T}$ is derivable in $\mathcal{T}'$ and vice-versa.

Equivalence of Theories

Definition (equality relation determined by a theory)

$\mathcal{T}$: Formal theory with some equations $X = Y$.

The equality relation determined by $\mathcal{T}$ is $=_{\mathcal{T}}$ is:

$$X =_{\mathcal{T}} Y \iff \mathcal{T} \vdash X = Y.$$

Extensionality in Lambda Calculus

Extensional Equality

- For functions: $\forall x\ (f(x) = g(x)) \Rightarrow f = g$.
- For programs: It two programs compute the same function, are they the same program?

Theorem

The theory $\lambda\beta$ is not extensional.

Proof.

Let $F \equiv y$ and $G \equiv \lambda x.yx$. Then for all X

$$\lambda\beta \vdash FX = GX,$$

but

$$\lambda\beta \nvdash F = G.$$

■

Extensional Equality

Rule and axiom-scheme to express extensionality

$$(\zeta) \quad \frac{Mx = Nx}{M = N} \qquad \text{if } x \notin \mathrm{FV}(MN),$$

$$(\eta) \quad \lambda x.Mx = M \qquad \text{if } x \notin \mathrm{FV}(M).$$

Definition (theories $\lambda\beta\zeta$ and $\lambda\beta\eta$)

$$\lambda\beta\zeta : \lambda\beta + (\zeta),$$
$$\lambda\beta\eta : \lambda\beta + (\eta).$$

Extensional Equality

Theorem

The theories $\lambda\beta\zeta$ and $\lambda\beta\eta$ are rule-equivalents.

Proof.

1. (ζ) is derivable in $\lambda\beta\eta$, i.e.
   $\lambda\beta\eta, Mx = Nx \vdash M = N$ (with $x \notin \mathrm{FV}(MN)$). (Whiteboard).
2. (η) is derivable in $\lambda\beta\zeta$, i.e.
   $\lambda\beta\zeta \vdash \lambda x.Mx = M$ (with $x \notin \mathrm{FV}(M)$). (Whiteboard). ∎

Definition (extensional equality in λ)

$$M =_{\lambda\mathsf{ext}} N \iff \lambda\beta\zeta \vdash M = N.$$

Beta-Eta Reduction in Lambda Calculus

Definition (η-redex and contractum)

An η-redex is any λ-term $\lambda x.Mx$ with $x \notin \mathrm{FV}(M)$. Its contractum is M.

Definition (η-contraction $(P \triangleright_{1\eta} Q)$)

Replace an occurrence of a η-redex in P by its contractum.

Definition (η-reduction $(P \triangleright_{\eta} Q)$)

P is changed to Q by a finite (perhaps empty) series of η-contractions and α-conversions.

Beta-Eta Reduction in Lambda Calculus

Definition ($\beta\eta$-redex)

An $\beta\eta$-redex is a β-redex or an η-redex.

Definition ($\beta\eta$-contraction ($P \triangleright_{1\beta\eta} Q$))

Replace an occurrence of a $\beta\eta$-redex in P by its contractum.

Definition ($\beta\eta$-reduction ($P \triangleright_{\beta\eta} Q$))

P is changed to Q by a finite (perhaps empty) series of $\beta\eta$-contractions and α-conversions.

Beta-Eta Reduction in Lambda Calculus

Definition ($\beta\eta$-normal forms)

A λ-term which contains no $\beta\eta$-redex.

Theorem (Church-Rosser theorem for $\triangleright_{\beta\eta}$)

$$\frac{P \triangleright_{\beta\eta} M \qquad P \triangleright_{\beta\eta} N}{\exists T.M \triangleright_{\beta\eta} T \wedge N \triangleright_{\beta} T}$$

Theorem (relation between $\triangleright_{\beta\eta}$ and $=_{\lambda\text{ext}}$)

$P =_{\lambda\text{ext}} Q$ iff Q can be obtained from P by a finite (perhaps empty) series of $\beta\eta$-contractions and reversed $\beta\eta$-contractions and α-conversions.

Beta-Eta Reduction in Lambda Calculus

Theorem (Church-Rosser theorem for $=_{\lambda\text{ext}}$)

$$\frac{P =_{\lambda\text{ext}} Q}{\exists T. P \rhd_{\beta\eta} T \land Q \rhd_{\beta\eta} T}$$

Corollary

The relation $=_{\lambda\text{ext}}$ is non-trivial (not all terms are $\beta\eta$-convertible to each other).

Extensionality in Combinatory Logic

Extensional Equality

Theorem

The theory CLw is not extensional.

Proof.

Let $X \equiv \mathsf{S}(\mathsf{K}u)\mathsf{I}$ and $Y \equiv u$, then for all M

$$\mathrm{CLw} \vdash XM = YM,$$

but

$$\mathrm{CLw} \nvdash X = Y.$$

Extensional Equality

Rule and axiom-scheme to express extensionality

$$(\zeta) \quad \frac{Xx = Yx}{X = Y} \qquad \text{if } x \notin \mathrm{FV}(XY),$$

$$(\xi) \quad \frac{X = Y}{[x].X = [x].Y}$$

$$(\eta) \quad [x].Ux = U \qquad \text{if } x \notin \mathrm{FV}(U).$$

Definition (theories $\mathrm{CL}\zeta$ and $\mathrm{CL}\xi$)

$$\mathrm{CL}\zeta : \mathrm{CL} + (\zeta),$$
$$\mathrm{CL}\xi : \mathrm{CL} + (\xi).$$

Extensional Equality

Exercise

Probe that neither (ζ) nor (ξ) are admissible in CLw (whiteboard).

Extensional Equality

Definition (extensional equality in CL)

$$X =_{C\text{ext}} Y \iff \mathrm{CL}\zeta \vdash X = Y.$$

Example

SK $=_{C\text{ext}}$ KI. (Whiteboard).

Extensional Equality

Theorem

The theory $\mathrm{CL}\xi$ determines the same equality-relation $=_{C\text{ext}}$ as $\mathrm{CL}\zeta$ does.

Proof.

1. (ζ) is derivable in $\mathrm{CL}\xi$, i.e.

   $\mathrm{CL}\xi, Xx = Yx \vdash X = Y$ (with $x \notin \mathrm{FV}(XY)$). (Whiteboard).

2. (ξ) is derivable in $\mathrm{CL}\zeta$, i.e.

   $\mathrm{CL}\zeta, X = Y \vdash [x].X = [x].Y$. (Whiteboard). ■

Axioms for Extensionality in CL

Definition (formal theory $\mathrm{CLext_{ax}}$)

$\mathrm{CLext_{ax}} : \mathrm{CLw} + \text{E-ax } 1 + \cdots + \text{E-ax } 5$, where

$$\mathsf{S}(\mathsf{S}(\mathsf{KS})(\mathsf{S}(\mathsf{KK})(\mathsf{S}(\mathsf{KS})\mathsf{K}))) = \mathsf{S}(\mathsf{KK}) \tag{E-ax 1}$$

$$\mathsf{S}(\mathsf{S}(\mathsf{KS})\mathsf{K})(\mathsf{KI}) = \mathsf{I} \tag{E-ax 2}$$

$$\mathsf{S}(\mathsf{KI}) = \mathsf{I} \tag{E-ax 3}$$

$$\mathsf{S}(\mathsf{KS})(\mathsf{S}(\mathsf{KK})) = \mathsf{K} \tag{E-ax 4}$$

$$\mathsf{S}(\mathsf{K}(\mathsf{S}(\mathsf{KS})))(\mathsf{S}(\mathsf{KS})(\mathsf{S}(\mathsf{KS}))) = \mathsf{S}(\mathsf{S}(\mathsf{KS})(\mathsf{S}(\mathsf{KK})(\mathsf{S}(\mathsf{KS})(\mathsf{S}(\mathsf{K}(\mathsf{S}(\mathsf{KS})))\mathsf{S}))))(\mathsf{KS}) \tag{E-ax 5}$$

Axioms for Extensionality in CL

Definition (other version of $\text{E-ax } 1, \ldots, \text{E-ax } 5$)

$$[x, y, v].\,(\mathsf{K}xv)(\mathsf{K}yv) = [x, y, v].xy \qquad \text{(E-ax 1)}$$

$$[x, v].\,(\mathsf{K}xv)(\mathsf{I}v) = [x, v].xv \qquad \text{(E-ax 2)}$$

$$[x, v].\,\mathsf{I}(xv) = [x, v].xv \qquad \text{(E-ax 3)}$$

$$[x, y, v].\,\mathsf{K}(xv)(yv) = [x, y, v].xv \qquad \text{(E-ax 4)}$$

$$[x, y, z, v].\,\mathsf{S}(xv)(yv)(zv) = [x, y, z, v].xv(zv)(yv(zv)) \qquad \text{(E-ax 5)}$$

Motivation

We are looking axioms which will make (ξ) admissible in $\mathrm{CLext}_{\mathrm{ax}}$:

$$\mathrm{CLext}_{\mathrm{ax}} \vdash X = Y \Longrightarrow \mathrm{CLext}_{\mathrm{ax}} \vdash [x].X = [x].Y.$$

Axioms for Extensionality in CL

Theorem

The theory $\mathrm{CLext}_{\mathrm{ax}}$ is theorem-equivalent to $\mathrm{CL}\xi$.

Strong Reduction

Definition (strong reduction $\succ$)

The formal theory of strong reduction:

Formulas: $X \succ Y$, where $X, Y \in \mathrm{CL}$-terms

Axiom-schemes and rules: The same than CLw changed '$=$' by '$\succ$' and the rule (σ) omitted.

New rule added:

$$(\xi)\ \frac{X \succ Y}{[x].X \succ [x].Y}$$

Example

SK $\succ$ KI. (Whiteboard).

Strong Reduction

Theorem (Church-Rosser theorem for $\succ$)

$$\frac{U \succ X \qquad U \succ Y}{\exists Z.X \succ Z \wedge Z \succ Y}$$

Definition (strong irreducibility)

X is called strongly irreducible iff, for all Y

$$X \succ Y \Longrightarrow Y \equiv X.$$

Theorem

The strongly irreducible CL-terms are exactly the terms in the strong nf class.

Models of CL

Applicative Structures

Definition (valuation)

Let D a set. A valuation is a mapping $\rho : \mathrm{Vars} \to D$.

Notation: $[d/x]\rho = \begin{cases} \rho(y), & \text{if } y \neq x; \\ d, & \text{otherwise.} \end{cases}$

Definition (applicative structure)

An applicative structure is a structure $\langle D, \cdot \rangle$ where

1. $|D| \geq 2$.
2. $\cdot : D^2 \to D$.

Combinatory Algebras

Definition (combinatory algebra)

A combinatory algebra is a structure (convention: association to the left for $\cdot$) $\mathbb{D} = \langle D, \cdot \rangle$ where

1. $|D| \geq 2$.
2. $\cdot : D^2 \to D$.
3. There are two elements $k, s \in D$ such that for all $a, b, c \in D$,

$$k \cdot a \cdot b = a, \tag{5}$$

$$s \cdot a \cdot b \cdot c = a \cdot c \cdot (b \cdot c). \tag{6}$$

Combinatory Algebras

Definition (model of CLw)

A model of CLw is a structure $\langle D, \cdot, i, k, s \rangle$ where

1. $\langle D, \cdot \rangle$ is a combinatory algebra.
2. The elements k and s satisfy (5) and (6).
3. The element i satisfies $i = s \cdot k \cdot k$.

Definition (model of $\mathrm{CLext}_{\mathrm{ax}}$)

A model of $\mathrm{CLext}_{\mathrm{ax}}$ is a model $\langle D, \cdot, i, k, s \rangle$ of CLw that satisfies the extensionality axioms $\text{E-ax } 1, \ldots, \text{E-ax } 5$.

Combinatory Algebras

Definition (interpretation of a term)

Let $\mathbb{D} = \langle D, \cdot, i, k, s\rangle$ where $\langle D, \cdot\rangle$ is a combinatory algebra and ρ a valuation. The interpretation of X in D under ρ, denoted $[\![X]\!]^{\mathbb{D}}_{\rho}$, is defined by

$$
\begin{aligned}
[\![X]\!]^{\mathbb{D}}_{\rho} &: \text{CL-term} \to D \\
[\![x]\!]^{\mathbb{D}}_{\rho} &= \rho(x), \\
[\![\mathsf{I}]\!]^{\mathbb{D}}_{\rho} &= i, \\
[\![\mathsf{K}]\!]^{\mathbb{D}}_{\rho} &= k, \\
[\![\mathsf{S}]\!]^{\mathbb{D}}_{\rho} &= s, \\
[\![XY]\!]^{\mathbb{D}}_{\rho} &= [\![X]\!]^{\mathbb{D}}_{\rho} \cdot [\![Y]\!]^{\mathbb{D}}_{\rho}.
\end{aligned}
$$

Combinatory Algebras

Definition (satisfaction)

$$\begin{aligned} \mathbb{D}, \rho \models X = Y &\iff [\![X]\!]^{\mathbb{D}}_{\rho} = [\![Y]\!]^{\mathbb{D}}_{\rho}, \\ \mathbb{D} \models X = Y &\iff (\forall \rho)(\mathbb{D}, \rho \models X = Y). \end{aligned}$$

Combinatory Algebras

Example (term model)

Let $\mathcal{T} \in \{\mathrm{CLw}, \mathrm{CLext}_{\mathrm{ax}}\}$. For each CL-term X,

$$[X] = \{Y : \mathcal{T} \vdash X = Y\}.$$

The $\mathbb{TM}(\mathcal{T})$ (the term model of $\mathcal{T}$) is $\langle D, \cdot, i, k, s\rangle$ where

$$\begin{aligned} D &= \{[X] : X \text{ is a } \mathrm{CL}\text{-term}\}, \\ [X] \cdot [Y] &= [XY], \\ i &= [\mathsf{I}], \\ k &= [\mathsf{K}], \\ s &= [\mathsf{S}]. \end{aligned}$$

Combinatory Algebras

Example (cont.)

In this model, interpretation is the same as substitution

$$[\![X]\!]_\rho = [[Y_1/x_1, \ldots, Y_n/x_n]X],$$

where

$$\begin{aligned} \mathrm{FV}(X) &= \{x_1, \ldots, x_n\}, \\ \forall x_i \in \mathrm{FV}(X).\rho(x_i) &= Y_i. \end{aligned}$$

Combinatory Algebras

Theorem (submodel theorem)

Let $\mathcal{T} \in \{\mathrm{CLw}, \mathrm{CLext}_{\mathrm{ax}}\}$. If $\langle D, \cdot, i, k, s\rangle$ is a model of $\mathcal{T}$ and D' is a subset of D which contains i, k and s and is closed under $\cdot$, then $\langle D', \cdot, i, k, s\rangle$ is a model of $\mathcal{T}$.

Definition (interiors)

Let $\mathcal{T} \in \{\mathrm{CLw}, \mathrm{CLext}_{\mathrm{ax}}\}$ and $\mathbb{D} = \langle D, \cdot, i, k, s\rangle$ a model of $\mathcal{T}$. The interior of $\mathbb{D}$ is

$$\mathbb{D}^\circ = \{[\![X]\!] : X \text{ closed}\}.$$

Theorem (interiors)

Let $\mathcal{T} \in \{\mathrm{CLw}, \mathrm{CLext}_{\mathrm{ax}}\}$. The interior of a model of $\mathcal{T}$ is also a model of $\mathcal{T}$.

Models of Lambda Calculus

The Definition of Lambda Model

Definition (λ-model)

A λ-model is a triple $\mathbb{D} = \langle D, \cdot, [\![\ ]\!] \rangle$ where

1. $\langle D, \cdot \rangle$ is an applicative structure.
2. $[\![\ ]\!]$: λ-terms $\to D$ is a mapping such that for each valuation ρ

$$
\begin{aligned}
[\![x]\!]_\rho &= \rho(x), \\
[\![PQ]\!]_\rho &= [\![P]\!]^{\mathbb{D}}_\rho \cdot [\![Q]\!]_\rho. \\
[\![\lambda x.P]\!]_\rho \cdot d &= [\![P]\!]_{[d/x]\rho}, && \text{for all } d \in D, \\
[\![M]\!]_\rho &= [\![M]\!]_\sigma && \text{if } \forall x \in \mathrm{FV} M.\rho(x) = \sigma(x), \\
[\![\lambda x.P]\!]_\rho &= [\![\lambda y.[y/x]P]\!]_\rho && \text{if } y \notin \mathrm{FV}(M),
\end{aligned}
$$

$$
\text{If } (\forall d \in D)([\![P]\!]_{[d/x]\rho} = [\![Q]\!]_{[d/x]\rho}) \text{ then } [\![\lambda x.P]\!]_\rho = [\![\lambda x.Q]\!]_\rho.
$$

The Definition of Lambda Model

Theorem

Every λ-model satisfies all the provable equations if the formal theory $\lambda\beta$.

The Definition of Lambda Model

Definition (models of $\lambda\beta\eta$)

A model of $\lambda\beta\eta$ is a λ-model that satisfies the equation $\lambda x.Mx = M$ for all terms M and all $x \notin \mathrm{FV}(M)$.

The Definition of Lambda Model

Example (term models)

Let $\mathcal{T} \in \{\lambda\beta, \lambda\beta\eta\}$. For each λ-term M,

$$[M] = \{N : \mathcal{T} \vdash M = N\}.$$

The $\mathbb{TM}(\mathcal{T})$ (the term model of $\mathcal{T}$) is $\langle D, \cdot, [\![\ ]\!] \rangle$ where

$$\begin{aligned} D &= \{[M] : M \text{ is a } \lambda\text{-term}\}, \\ [P] \cdot [Q] &= [PQ], \\ [\![M]\!]_\rho &= [[N_1/x_1, \dots, N_n/x_n]M], \end{aligned}$$

where

$$\begin{aligned} \mathrm{FV}(M) &= \{x_1, \dots, x_n\}, \\ \forall x_i \in \mathrm{FV}(M). \rho(x_i) &= N_i. \end{aligned}$$

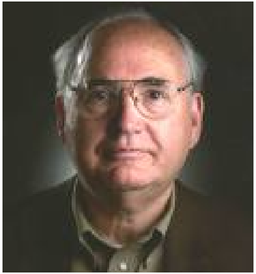
Scott's D_∞: Complete Partial Orders

The problem

> *"In the λ-calculus the objects serve both as arguments and as functions to be applied to these arguments. Therefore one would like that a semantics for λ-calculus consist of a domain D such that its function space D^D is isormorphic to D. By Cantor's theorem this is impossible." [Barendregt (1981) 2004, p. 86]*

Scott's D_∞: Complete Partial Orders

Solution



Dana Scott

- D_∞: complete partial order
- $[D_\infty \to D_\infty]$: continuous functions (under Scott's topology)
- $D_\infty \cong [D_\infty \to D_\infty]$.

Scott's D_∞: Complete Partial Orders

Definition (partially ordered sets (poset))

A poset is a structure $\langle D, \sqsubseteq \rangle$ where D is a set and $\sqsubseteq$: $D^2 \to D$ is transitive, anti-symmetric, and reflexive.

Let $\langle D, \sqsubseteq \rangle$ a poset and let $X \subseteq D$.

Definition (upper bound)

An upper bound (u.b) of X is any $b \in D$ such

$$\forall a \in X. a \sqsubseteq b.$$

Definition (least upper bound (l.u.b.) (or supremum))

The l.u.b. of X called $\bigsqcup X$, it is an upper bound b of X such

$$\forall c \in D. c \text{ is a u.b. of } X \Longrightarrow b \sqsubseteq c.$$

Scott's D_∞: Complete Partial Orders

Definition (bottom)

D has an element called bottom (denoted $\bot$) iff

$$\forall x \in D. \bot \sqsubseteq x.$$

Definition (directed sets)

Let $\langle D, \sqsubseteq \rangle$ a poset. A subset $X \subseteq D$ is said to be directed iff $X \neq \emptyset$ and

$$\forall a, b \in X. \exists c \in X. a \sqsubseteq c \wedge b \sqsubseteq c.$$

Definition (complete partial orders, c.p.o.s)

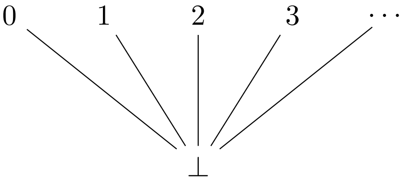
A c.p.o. is a poset $\langle D, \sqsubseteq \rangle$ such that

1. D has a $\bot$.
2. Every direct subset $X \subseteq D$ has a l.u.b.

Scott's D_∞: Complete Partial Orders

Definition (set $\mathbb{N}^+$)

$$\mathbb{N}^+ = \mathbb{N} \cup \{\bot\} \quad (\bot \notin \mathbb{N}),$$
$$\forall a, b \in \mathbb{N}^+ . a \sqsubseteq b \iff (a = \bot \land b \in \mathbb{N}) \lor a = b.$$



- The element $\bot$ represents an undefined value (partial functions).
- $a \sqsubseteq b$ represents that b "is more defined" than a or both are equals (semantic approximation order).

Theorem

$\langle \mathbb{N}^+, \sqsubseteq \rangle$ is a c.p.o.

Scott's D_∞: Complete Partial Orders

Let $\langle D, \sqsubseteq \rangle$ and $\langle D', \sqsubseteq' \rangle$ be c.p.o.s and φ a function $\varphi : D \to D'$.

Definition (monotonicity)

The function φ is monotonic iff

$$a \sqsubseteq b \Longrightarrow \varphi(a) \sqsubseteq' \varphi(b).$$

Example

Let $\varphi : \mathbb{N}^+ \to \mathbb{N}^+$ be a monotonic function. If $\varphi(\perp) = 1$ then φ is a constant function, i.e. $\forall n \in \mathbb{N}^+ . \varphi(n) = 1$.

Scott's D_∞: Complete Partial Orders

Definition (continuity)

The function φ is continua iff, for all directed $X \subseteq D$

$$\varphi(\bigsqcup X) = \bigsqcup(\varphi(X)),$$

where

$$\varphi(X) = \{\varphi(a) : a \in X\}.$$

Scott's D_∞: Complete Partial Orders

Definition (function-set $[D \to D']$)

$[D \to D']$: For c.p.o.s $\langle D, \sqsubseteq \rangle$ and $\langle D', \sqsubseteq' \rangle$, the set of all continuous functions.
For $\varphi, \psi \in [D \to D']$, we define

$$\varphi \sqsubseteq \psi \iff \forall d \in D. \varphi(d) \sqsubseteq' \psi(d).$$

Theorem

The function $\forall d \in D. \bot(d) = \bot'$ is the bottom of $[D \to D']$.

Theorem

$[D \to D']$ is a c.p.o.

Scott's D_∞: The Construction

Definition (sequence $D_0, D_1, \dots$)

$$D_0 = \mathbb{N}^+,$$
$$D_{n+1} = [D_n \to D_n].$$

Theorem

Every D_n is a c.p.o.

Scott's D_∞: The Construction

Example

From: http://en.wikibooks.org/wiki/Haskell/Denotational_semantics
The factorial function

$$f(n) = \text{ if } n == 0 \text{ then } 1\text{else } n \cdot f(n-1)$$

Approximations of the factorial function

$$f_{k+1}(n) = \text{ if } n == 0 \text{ then } 1 \text{ else } n \cdot f_k(n-1)$$

Scott's D_∞: The Construction

Example (cont.)

$$f_0(n) = \bot \,, \qquad f_1(n) = \begin{cases} 1 & \text{if } n \text{ is } 0 \\ \bot & \text{else} \end{cases} \,,$$

$$f_2(n) = \begin{cases} 1 & \text{if } n \text{ is } 0 \\ 1 & \text{if } n \text{ is } 1 \\ \bot & \text{else} \end{cases} \,, \; f_3(n) = \begin{cases} 1 & \text{if } n \text{ is } 0 \\ 1 & \text{if } n \text{ is } 1 \\ 2 & \text{if } n \text{ is } 2 \\ \bot & \text{else} \end{cases} \,, \ldots$$

Then, $\bot = f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$.
The idea is

$$\forall n. \bigsqcup (f_0(n) \sqsubseteq f_1(n) \sqsubseteq f_2(n) \sqsubseteq \ldots) = f(n).$$

Scott's D_∞: The Construction

About the λ-model $\langle D_\infty, \cdot, [\![\ ]\!] \rangle$

- D_∞ cannot be a set of functions (no function can be applied to itself).
- Scott's idea:
  - Members of D_∞ are infinite sequences of functions

$$\varphi = \langle \varphi_0, \varphi_1, \varphi_2, \ldots \rangle, \text{ where } \varphi_n \in D_n.$$

  - Application

$$\varphi \cdot \psi = \langle \varphi_1(\psi_0), \varphi_2(\psi_1), \ldots \rangle$$

  - Self-application

$$\varphi \cdot \varphi = \langle \varphi_1(\varphi_0), \varphi_2(\varphi_1), \ldots \rangle$$

References

Barendregt, H. P. [1981] (2004). The Lambda Calculus. Its Syntax and Semantics. Revised edition, 6th impression. Vol. 103. Studies in Logic and the Foundations of Mathematics. Elsevier (cit. on pp. 3, 110).

Barendregt, Henk and Barendsen, Erik (2000). Introduction to Lambda Calculus. Revisited edition (cit. on p. 3).

Hindley, J. Roger and Seldin, Jonathan P. (2008). Lambda-Calculus and Combinators. An Introduction. Cambridge University Press (cit. on pp. 3, 4).

Peyton Jones, Simon L. (1987). The Implementation of Functional Programming Languages. Series in Computer Sciences. Prentice-Hall International (cit. on p. 62).