Hipercomputación desde la computación cuántica y la tesis de Church-Turing

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Temas

Introducción

Algoritmos cuánticos y ciencias de la computación

- Complejidad algorítmica
  - Shor (1994): Factorizar un número en sus factores primos
  - Grover (1996): Busqueda en una base de datos desorganizada
- Computabilidad
  Posibilidad de “computar lo incomputable”

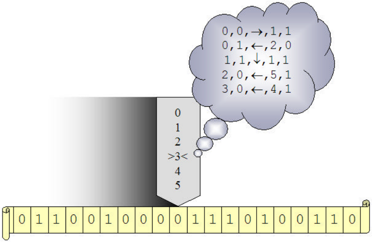
Máquinas de Turing

1936: el año de la fundamentación

máquinas de Turing $\equiv$ máquinas de Post $\equiv$ λ-cálculo $\equiv$ funciones recursivas

Alan Turing

Ejemplo



Tesis de Church-Turing I

Definición Church (1936)

“*We now define the notion, already discussed, of an effectively calculable function of positive integers by identifying it with the notion of a recursive function of positive integers (or of a λ-definable function of positive integers). This definition is thought to be justified by considerations which follow, so far as positive justification can ever be obtained for the selection of a formal definition to correspond to intuitive notion.*”[a]

[a] A. Church. An unsolvable problem of elementary number theory. American Journal of Mathematics, vol 58(2), p. 356, 1936.

Tesis de Church-Turing II

Definición de Turing (1936)

“*The “computable” numbers include all numbers which would naturally be regarded as computable.*”[a]

[a] A. M. Turing. On computable numbers, with an application to the Entscheidungsproblem. Proc. London Math. Soc., vol 42, p. 249, 1936-7.

Tesis de Church-Turing III

Evolución de la terminología (Kleene)

Kleene (1943)[a]: Definición de Church $\equiv$ Thesis I
Kleene (1952)[b]: Tesis de Church y tesis de Turing
Kleene (1967)[c]: Tesis de Church-Turing

[a] S. C. Kleene. Recursive predicates and quantifiers. Transactions of the American Mathematical Society, vol 53, p. 60, 1943.

[b] S. C. Kleene. Introduction to metamathematics. Wolters-Noordhoff Publishing. Groningen, 1952.

[c] S. C. Kleene. Mathematical Logic. New York: Wiley, 1967

Tesis de Church-Turing IV

Tesis de Church-Turing

"*Any procedure than can be carried out by an idealised human clerk working mechanically with paper and pencil can also be carried out by a Turing machine.*"

Una función es efectivamente calculable si y sólo si es computable por una máquina de Turing

Posibles refutaciones

1. $\Longleftarrow$
2. $\Longrightarrow$

Tesis de Church-Turing V

Otras “versiones” de la tesis de Chuch-Turing

Complejidad algorítmica: *Any “reasonable” (in principle physically realizable) model of computation can be efifciently simulated on a probabilistic Turing machine*.

Física: *Whatever physical system can compute just recursive functions*.

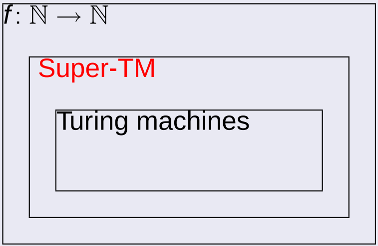
Máximal: *Whatever can be calculated by a machine (working on finite data in accordance with a finite program of instructions) is Turing machine computable*.

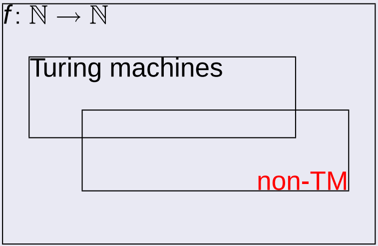
Hypercomputation I

Definition

A hypercomputer is any machine (theoretical or real) that compute functions or numbers, or more generally solve problems or carry out tasks, that cannot be computed or solved by a Turing machine (TM).

Types





Hypercomputation II

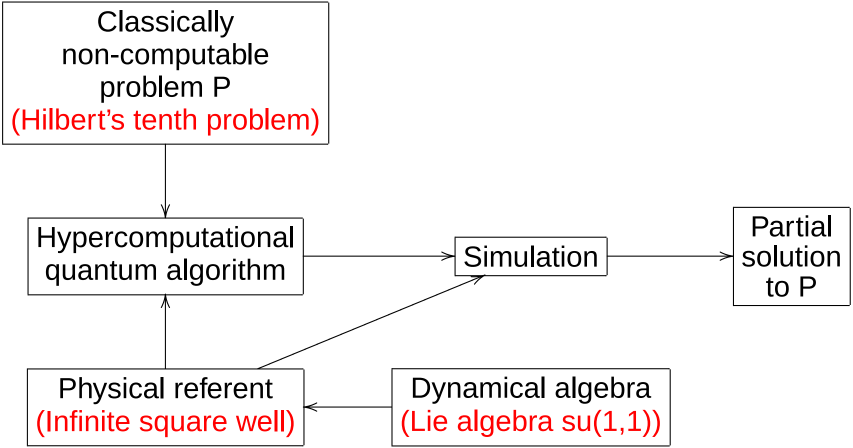
Examples

- Oracle Turing Machine (Turing)
- Accelerating Turing machine (Copeland)
- Analog Recurrent Neural Network (Sielgelmann and Stong)
- ⋮

Implementation

The possibility of real construction of a hypercomputer is controversial and is still under analysis.

Key ideas: Hypercomputational quantum algorithm à la Kieu[1]



[1] T. D. Kieu. Hypercomputability of quantum adiabatic processes: Fact versus prejudices. quant-ph/0504101, 2005.

Incomputable-(Turing Machine) problem

Hilbert's tenth problem

Given a Diophantine equation

$$D(x_1, \ldots, x_k) = 0 \ ,$$

we should build a procedure to determine whether or not this equation has a solution in $\mathbb{N}$.

Classical solution (Matiyasevich, Davis, Robinson, Putnam)

Hilbert's tenth problem $\equiv$ Halting problem for Turing Machine

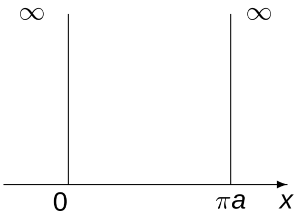
Physical referent: Infinite Square Well I

- Quantum system: particle with mass m trapped inside the infinite square well $0 \leq x \leq \pi a$

$$V(x) = \begin{cases} 0 \equiv \frac{\hbar^2}{2ma^2}, & \text{if } x \in (0, \pi a) \ , \\ \infty, & \text{otherwise} \ , \end{cases}$$

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar^2}{2ma^2} \ ,$$

$$\psi(x) = 0, \quad x \geq \pi a \text{ and } x \leq 0 \ ,$$



Physical referent: Infinite Square Well II

- Computational basis and action of H on it:

$$\{|n\rangle \mid n \in \mathbb{N}\} \quad ,$$
$$H|n\rangle = E_n|n\rangle\,, \text{ where } E_n = (\hbar^2/2ma^2)n(n+2) \quad .$$

- Dynamical Lie algebra $\mathfrak{su}(1,1)$ associated with ISW:

$$[K_-, K_+] = K_3 \quad , \qquad [K_\pm, K_3] = \mp 2K_\pm \quad .$$

- Infinite-dimensional irreducible representation for $\mathfrak{su}(1,1)$

$$K_+|n\rangle = \sqrt{(n+1)(n+3)}\,|n+1\rangle \text{ (creation operator)} \quad ,$$
$$K_-|n\rangle = \sqrt{n(n+2)}\,|n-1\rangle \text{ (annihilation operator)} \quad ,$$
$$K_3|n\rangle = (2n+3)\,|n\rangle \text{ (Cartan operator)} \quad .$$

Physical referent: Infinite Square Well III

- Number operator

$$N = (1/2)(K_3 - 3) = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad ,$$

$$N \,|\, n\rangle = n \,|\, n\rangle \quad .$$

- Barut-Girardello coherent states ($K_- \,|\, z\rangle = z \,|\, z\rangle$, with $z \in \mathbb{C}$):

$$|\, z\rangle = \frac{|z|}{\sqrt{I_2(2\,|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!(n+2)!}} \,|\, n\rangle \quad .$$

Algorithm's strategics I

Codification *à la* Kieu[a]

$$\begin{array}{ccc} D(x_1,\ldots,x_k)=0 & \xrightarrow{\text{codification}} & H_{\mathrm{D}}=(D(N_1,\ldots,N_k))^2 \\ \downarrow & & \downarrow \\ \text{Solution in } \mathbb{N} & \xrightarrow{\text{if and only if}} & H_{\mathrm{D}}\,|\,\{n\}\rangle_0=0 \end{array}$$

Note: Continuos quantum computation

[a] A. Sicard, M. Vélez, and J. Ospina. A possible hypercomputational quantum algorithm. In E. J. Donkor et all. editors, Quantum Information and Computation III, vol. 5815 of Proc. of SPIE. Preprint `quant-ph/0406137`. Forthcoming.

Algorithm's strategics II

New problem

To find the ground state $|\{n\}\rangle_0$ of $H_{\rm D}$.

Solution *à la* Kieu: adiabatic quantum computation

$$H_{\rm A}(t) = (1 - t/T)H_{\rm I} + (t/T)H_{\rm D} \text{ over } t \in [0, T]$$

$$H_{\rm I} = \sum_{i=1}^{k} (K_{+_i} - z_i^*)(K_{-i} - z_i)\,, \quad |\psi(0)\rangle = \bigotimes_{i=1}^{k} |z_i\rangle \quad .$$

Hypercomputational Quantum Algorithm

1. Construct a physical process subject to $H_{\mathrm{A}}(t)$ over the time interval $[0, T]$, for some finite time T.
2. Measure through the time-dependent Schrödinger equation $i\partial_t \,|\, \psi(t)\rangle = H_{\mathrm{A}}(t) \,|\, \psi(t)\rangle$, for $t \in [0, T]$ the maximum probability

$$P_{\max}(T) = \max_{(n_1,\ldots,n_k)\in\mathbb{N}^k} |\langle\psi(T) \mid n_1,\ldots,n_k\rangle|^2 = \left|\langle\psi(T) \mid \{n\}\rangle_0\right|^2 .$$

3. If $P_{\max}(T) \leq 1/2$, increase T and repeat all the steps above .
4. If $P_{\max}(T) > 1/2$ (halting criterion) then $|\,\{n\}\rangle_0$ is the ground state of H_{D} (assuming no degeneracy).
5. $D(x_1, \ldots, x_k) = 0$ has a solution in $\mathbb{N}$, if and only if, $H_{\mathrm{D}} \,|\, \{n\}\rangle_0 = 0$.

Conclusiones

- La hipercomputación y la tesis de Church-Turing son compatibles:
  - La tesis de Church-Turing no impide la existencia de modelos de hipercomputación
  - Los modelos de hipercomputación (conocidos hasta el momento), no refutan la tesis de Church-Turing
- Los modelos de hipercomputación desde la computación cuántica refutan la versión débil de la tesis máximal:
  - Refutación fuerte (máquina real): Problema abierto
  - Refutación débil (máquina teórica): Refutada

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