Dedekind-Peano Arithmetic from Set Theory

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Introduction: Uses

Natural numbers have been used for

(i) counting (cardinal numbers),

(ii) ordering (ordinal numbers).

Timeline: `https://mathigon.org/timeline`.

Introduction: Kronecker's Quote

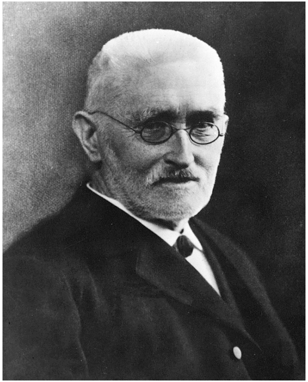
Leopold Kronecker (1823 – 1891)
(image from Wikipedia)

'*Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.*' [Weber 1893, p. 15]

'God made the integers, and all the rest is the work of man.' [Merzcbach and Boyer (1968) 2011, p. 542]

Towards an Axiomatisation of the Arithmetic

Hermann Grassmann
(1809 – 1877)



Richard Dedekind
(1831 – 1916)

Giuseppe Peano
(1858 – 1932)

(Images from Wikipedia)

Towards an Axiomatisation of the Arithmetic

Publications timeline (incomplete)

- Grassmann 1861. *Lehrbuch der Mathematik für höhere Lehranstalten*. (Mathematics Textbook for Higher Educational Institutions).
- Dedekind 1888. *Was sind und was sollen die Zahlen?* (What are numbers and what should they be?)
- Peano 1889. *Arithmetices Principia: Nova Methodo Exposita*. (The Principles of Arithmetic, Presented by a New Method)
- Dedekind 1890. Letter to Keferstein.

Towards an Axiomatisation of the Arithmetic

Peano's Axioms for Arithmetic

Original version [Peano (1889) 1967, p. 94].

The sign N means *number* (*positive integer*).
The sign 1 means *unity*.
The sign $a + 1$ means *the successor of* a, or a *plus* 1.
The sign $=$ means *is equal to*. We consider this sign as new, although it has the form of a sign of logic.

Axioms

1. $1 \,\varepsilon\, \mathrm{N}.$
2. $a \,\varepsilon\, \mathrm{N} \,.\!\supset\!.\, a = a.$
3. $a, b \,\varepsilon\, \mathrm{N} \,.\!\supset\!:\, a = b \,.\!=\!.\, b = a.$
4. $a, b, c \,\varepsilon\, \mathrm{N} \,.\!\supset\!\therefore\, a = b \,.\, b = c \,:\!\supset\!.\, a = c.$
5. $a = b \,.\, b \,\varepsilon\, \mathrm{N} \,:\!\supset\!.\, a \,\varepsilon\, \mathrm{N}.$
6. $a \,\varepsilon\, \mathrm{N} \,.\!\supset\!.\, a + 1 \,\varepsilon\, \mathrm{N}.$
7. $a, b \,\varepsilon\, \mathrm{N} \,.\!\supset\!:\, a = b \,.\!=\!.\, a + 1 = b + 1.$
8. $a \,\varepsilon\, \mathrm{N} \,.\!\supset\!.\, a + 1 \,-\!= 1.$
9. $k \,\varepsilon\, \mathrm{K} \therefore 1 \,\varepsilon\, k \therefore x \,\varepsilon\, \mathrm{N} \,.\, x \,\varepsilon\, k \,:\!\supset_x\!.\, x + 1 \,\varepsilon\, k \,::\!\supset\!.\, \mathrm{N} \supset k.$

Modern version [Wang 1957, p. 149].

The basic concepts are: 1, number, successor.
The axioms are:

P1. 1 is a number.

P2. The successor of any number is a number.

P3. No two numbers have the same successor.

P4. 1 is not the successor of any number.

P5. Any property which belongs to 1, and also to the successor of every number which has the property, belongs to all numb

Foundations of Mathematics

Some foundational systems*

(i) Set theories
(ii) Category theories
(iii) Type theories
(iv) Univalent foundations
(v) Homotopy type theories

*See, for example, [Centrone, Kant and Sarikaya 2019].

First-order logic: Two historical remarks

(i) ‘First-order logic was explicitly identified by Peirce in 1885, but then forgotten. It was independently re-discovered in Hilbert’s 1917/18 lectures, and given wide currency in the 1928 monograph, Hilbert & Ackermann. Peirce was the first to identify it: but it was Hilbert who put the system on the map.’ [Ewald 2019]

First-Order Theories

First-order logic: Two historical remarks

(i) 'First-order logic was explicitly identified by Peirce in 1885, but then forgotten. It was independently re-discovered in Hilbert's 1917/18 lectures, and given wide currency in the 1928 monograph, Hilbert & Ackermann. Peirce was the first to identify it: but it was Hilbert who put the system on the map.' [Ewald 2019]

(ii) 'Nevertheless, Hilbert did not at any point regard first-order logic as the proper basis for mathematics... It was in Skolem's work on set theory (1923) that first-order logic was first proposed as all of logic and that set theory was first formulated within first-order logic.' [Moore 1988, p. 128]

First-Order Theories

Preliminaries logics

- First-order logic with identity
- Non-logic symbols and non-logic axioms
- Theories
- Definitions

First-Order Theories

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- Non-logic symbols and non-logic axioms
- Theories
- Definitions

Example

Group theory.

First-Order Dedekind-Peano Arithmetic

Non-logical symbols

The formal language $\mathfrak{L}$ of the first-order theory of arithmetic (FA) is defined by

$$\mathfrak{L} = \{', +, *, 0\}, \quad \text{where}$$

(i) the symbol $'$ is a unary function symbol (successor function),

(ii) the symbol $+$ is a binary function symbol (addition function),

(iii) the symbol $*$ is a binary function symbol (multiplication function) and

(iv) the symbol 0 is a constant symbol (zero element).

First-Order Dedekind-Peano Arithmetic

Axioms

Non-logical axioms of FA.*

$$\forall n\,(0 \neq n') \qquad (\mathsf{FA}_1)$$

$$\forall m \forall n\,(m' = n' \supset m = n) \qquad (\mathsf{FA}_2)$$

$$\forall n\,(n + 0 = n) \qquad (\mathsf{FA}_3)$$

$$\forall m \forall n\,(m + n' = (m + n)') \qquad (\mathsf{FA}_4)$$

$$\forall n\,(n * 0 = 0) \qquad (\mathsf{FA}_5)$$

$$\forall m \forall n\,(m * n' = (m * n) + m) \qquad (\mathsf{FA}_6)$$

For any property P,

$$P0 \wedge \forall n\,(Pn \supset P(n')) \supset \forall n Pn \qquad (\mathsf{FA}_7) \text{ (axiom schema of induction)}$$

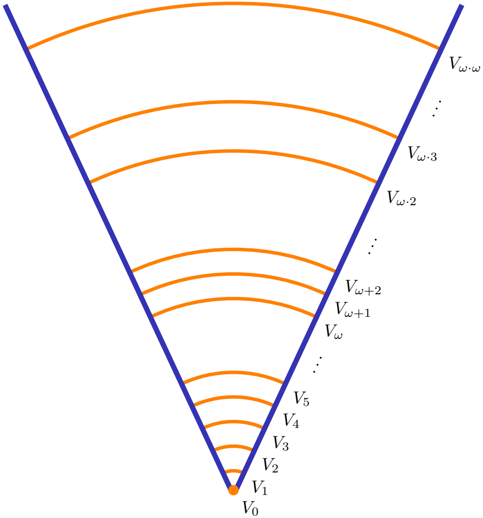
*See, for example, [Machover 1996; Hájek and Pudlák (1993) 1998; Skolem 1955; Robinson 1949].

Set Theories as Foundations

Some axiomatic set theories

- Zermelo-Fraenkel set theory (ZF)
- Zermelo-Fraenkel set theory with Choice (ZFC)
- von Neumann-Bernays-Gödel set theory (NBG)
- Morse-Kelley set theory (MK)
- Tarski-Grothendieck set theory (TG)

von Neumann Hierarchy of Sets



$$
\begin{aligned}
V_0 &:= \emptyset \\
V_{n+1} &:= \mathcal{P} V_n \\
&\vdots \\
V_\omega &:= \bigcup_{n<\omega} V_n \\
V_{\omega+1} &:= \mathcal{P} V_\omega \\
&\vdots \\
V_\lambda &:= \bigcup_{\beta<\lambda} V_\beta \\
&\vdots
\end{aligned}
$$

Ti*k*Z image adapted from `https://tex.stackexchange.com/a/635569`.

Defining Natural Numbers from Set Theory

Definitional (non-axiomatic) approach

- We shall define natural numbers in terms of sets.
- We shall prove the properties of natural numbers from properties of sets.

Defining Natural Numbers from Set Theory

von Neumann's construction

Informally: A natural number is the set of all smaller natural numbers (impredicative definition).

$$
\begin{aligned}
0 &:= \emptyset, \\
1 &:= \{0\} &&= \{\emptyset\}, \\
2 &:= \{0, 1\} &&= \{\emptyset, \{\emptyset\}\}, \\
3 &:= \{0, 1, 2\} &&= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\vdots
\end{aligned}
$$

Defining Natural Numbers from Set Theory

Definition

Let a be a set. The **successor** of a is

$$a^+ := a \cup \{a\}.$$

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Example

$$\begin{aligned} 0 &= \emptyset, \\ 1 &= \emptyset^{+}, \\ 2 &= \emptyset^{++}, \\ 3 &= \emptyset^{+++}, \\ &\vdots \end{aligned}$$

Zermelo-Fraenkel Set Theory with Choice (ZFC)

Ernst Zermelo
(1871 – 1853)

Adolf Fraenkel
(1891 – 1965)

(Images from Wikipedia)

Zermelo-Fraenkel Set Theory with Choice (ZFC)

Thoralf Skolem
(1887 – 1863)

John von Neumann
(1903 – 1957)

(Images from Wikipedia)

Zermelo-Fraenkel Set Theory with Choice (ZFC)

ZFC as a foundational system for mathematics

- 'Our axioms provide a sufficient collection of assumptions for the development of the whole of mathematics—a remarkable fact.' [Enderton 1977, p. 11]
- 'Experience has shown that practically all notions used in contemporary mathematics can be defined, and their mathematical properties derived, in this axiomatic system. In this sense, the axiomatic set theory serves as a satisfactory foundations for the other branches of mathematics.' [Hrbacek and Jech (1978) 1999, p. 3]
- 'Conventional mathematics is based on ZFC (the Zermelo-Fraenkel axioms, including the Axiom of Choice). Working withing ZFC, on develops:... All the mathematics found in basic texts on analysis, topology, algebra, etc.' [Kunen (2011) 2013, p. 1]

Zermelo-Fraenkel Set Theory with Choice (ZFC)

Primitive notions

We only need two primitive notions, ‘set’ and ‘member’.

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Non-logical symbols

In our formalisation of ZFC, the set of non-logical symbols is

$$\mathfrak{L} = \{\epsilon\},$$

where ϵ is a binary predicate (relation) symbol.

ZFC Axioms

Extensionality axiom

If two sets have exactly the same members, then they are equal, that is,

$$\forall A \, \forall B \, [\forall x \, (x \in A \leftrightarrow x \in B) \rightarrow A = B \,].$$

ZFC Axioms

Extensionality axiom

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$$\forall A \, \forall B \, [\, \forall x \, (x \in A \leftrightarrow x \in B) \rightarrow A = B \,].$$

Question

Have we any set? No, we haven't.

ZFC Axioms

Empty set axiom

There is a set having no members, that is,

$$\exists B \, \forall x \, (x \notin B).$$

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Remark

The empty set axiom is equivalent to

$$\exists B \, \forall x \, (x \in B \leftrightarrow x \neq x).$$

ZFC Axioms

Pairing axiom

For any sets u and v, there is a set having as members just u and v, that is,

$$\forall a\,\forall b\,\exists C\,\forall x\,(x \in C \leftrightarrow x = a \vee x = b).$$

ZFC Axioms

Union axiom

For any sets a and b, there is a set whose members are those sets belonging either to a or to b (or both), that is,

$$\forall a\, \forall b\, \exists B\, \forall x\, (x \in B \leftrightarrow x \in a \vee x \in b).$$

ZFC Axioms

Power set axiom

For any set a, there is a set whose members are exactly the subsets of a, that is,

$$\forall a \, \exists B \, \forall x \, (x \in B \leftrightarrow x \subseteq a),$$

where

$$u \subseteq v := \forall t \, (t \in u \to t \in v).$$

Definitions from Set Abstraction

Definitions from the empty, pairing, union and power set axioms via set abstraction

Let a, b, u and v be sets, then we define

$$
\begin{aligned}
\emptyset &:= \{\, x \mid x \neq x \,\} && \text{(empty set)},\\
\{u, v\} &:= \{\, x \mid x = u \lor x = v \,\} && \text{(pair set)},\\
\{u\} &:= \{u, u\} && \text{(singleton set)},\\
a \cup b &:= \{\, x \mid x \in a \lor x \in b \,\} && \text{(union)},\\
\mathcal{P}a &:= \{\, x \mid x \subseteq a \,\} && \text{(power set)}.
\end{aligned}
$$

ZFC Axioms

Subset axiom scheme (axiom scheme of comprehension, axiom scheme of separation)

For any propositional function $\varphi(x)$, not containing B, the following is an axiom:

$$\forall c \, \exists B \, \forall x \, (x \in B \leftrightarrow x \in c \wedge \varphi(x)).$$

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We stated an axiom scheme.

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Set abstraction from the subset axiom scheme

$\{\, x \in c \mid \varphi(x) \,\}$ is the set of all $x \in c$ satisfying the property φ.

ZFC Axioms

Definition

A set A is **inductive** iff

- $\emptyset \in A$ and
- if $a \in A$ then $a^+ \in A$.

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- if $a \in A$ then $a^+ \in A$.

Remark

An inductive is an infinite set.

Question

Are there inductive sets?

ZFC Axioms

Infinity axiom

There exists an inductive set, that is,

$$\exists A\,[\,\emptyset \in A \land \forall a\,(a \in A \to a^{+} \in A)\,].$$

The Set of Natural Numbers

Definition

A **natural number** is a set that belongs to every inductive set.

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Theorem

There is a set whose members are exactly the natural numbers [Enderton 1977, Theorem 4A].

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Proof.

Let A be an inductive set. By the subset axiom scheme, there is a set

$$\{\, x \in A \mid x \in I \text{ for every inductive set } I \,\}.$$

∎

The Set of Natural Numbers

Definition

The set of **all natural numbers**, denoted by ω, is defined by

$$\omega := \{\, x \in A \mid x \in I \text{ for every inductive set } I \,\}.$$

That is,

$$x \in \omega \quad \text{iff} \quad x \text{ is a natural number.}$$

The Set of Natural Numbers

Theorem

The set ω is inductive, and it is a subset of every other inductive set [Enderton 1977, Theorem 4B].

The Set of Natural Numbers

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Remark

The set w is the smallest inductive set

The Set of Natural Numbers

Remark

Since that the collection of all inductive sets is not a set but a proper class, using class we could define the set of natural numbers by

$$\omega := \bigcap \{ A \mid A \text{ is an inductive set} \}.$$

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Remark

Mendelson [(1973) 2008] in the proof of Theorem ZFC 8 defines the set ω as an intersection of some inductive sets.

Induction Principle for Natural Numbers

Induction principle for ω

Any inductive subset of ω coincides with ω [Enderton 1977, p. 69].

Induction Principle for Natural Numbers

Induction principle for ω (other version)

Let $P(x)$ be a property. Assume that

(i) $P(0)$ holds,

(ii) for all $n \in \omega$, $P(n)$ implies $P(n^+)$.

Then P holds for all natural numbers n [Hrbacek and Jech (1978) 1999].

Proof.

'This is an immediate consequence of our definition of w. The assumptions (i) and (ii) simple say that the set $A = \{\, n \in \omega \mid P(n) \,\}$ is inductive. $\omega \subseteq A$ follows.' [Hrbacek and Jech (1978) 1999, p. 42] ■

Recursion on Natural Numbers

Recursion theorem on ω

Let A be a set, $a \in A$ and $F : A \to A$. Then there exists a unique function h such that [Enderton 1977, p. 73]

$$\begin{aligned} h : \omega \to A \\ h(0) &= a, \\ h(n^+) &= F(h(n)), \text{ for all } n \in \omega. \end{aligned}$$

Arithmetic

Idea

We shall apply the recursion theorem to define addition and multiplication on ω.

Arithmetic

Example

We want to define the function

$$A_5 : w \to w := n \mapsto \text{addition of } 5 \text{ to } n.$$

Arithmetic

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$$A_5 : w \to w := n \mapsto \text{addition of } 5 \text{ to } n.$$

Let $F : \omega \to \omega := n \mapsto n^+$. By the recursion theorem there exists a unique function

$$\begin{aligned} A_5 : w \to w \\ A_5(0) = 5, \\ A_5(n^+) = (A_5(n))^+. \end{aligned}$$

Arithmetic

Example

Let $m \in \omega$. By the recursion theorem there exists a unique function

$$A_m : w \to w$$
$$A_m(0) = m,$$
$$A_m(n^+) = (A_m(n))^+.$$

Arithmetic

Definition

Let m and n be natural numbers. We define the **addition** of m and n by

$$(+) : w \times w \to w$$
$$m + n = A_m(n).$$

Arithmetic

Theorem

Let m and n be natural numbers. Then

$$\begin{aligned} n + 0 &= n, \\ m + n^+ &= (m + n)^+. \end{aligned}$$

Arithmetic

Example

Let $m \in \omega$. By the recursion theorem there exists a unique function

$$
\begin{aligned}
M_m : w \to w \\
M_m(0) &= 0, \\
M_m(n^+) &= M_m(n) + m.
\end{aligned}
$$

Arithmetic

Definition

Let m and n be natural numbers. We define the **multiplication** of m and n by

$$(\cdot) : w \times w \to w$$
$$m \cdot n = M_m(n).$$

Arithmetic

Theorem

Let m and n be natural numbers. Then

$$\begin{aligned} n \cdot 0 &= 0, \\ m \cdot n^+ &= (m \cdot n) + m. \end{aligned}$$

First-Order Dedekind-Peano Arithmetic from ZFC

Done!

(i) Zero ✓

(ii) Successor ✓

(iii) Addition ✓

(iv) Multiplication ✓

(v) Axiom schema of induction ✓

Final Comments

(i) Benacerraf's identification problem: Problem in reducing natural numbers to pure sets [Benacerraf 1965].

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(ii) Formalisation of mathematics: An error-prone task

- In Principia Mathematica, Whitehead and Russell's magnum opus, the proof that $1 + 1 = 2$ is in page 360 (see Wikipedia).
- In the mathematics of Bourbaki, the definition of number 1 requires approximately 4.5×10^{12} symbols [Mathias 2002].

Final Comments

(i) Benacerraf's identification problem: Problem in reducing natural numbers to pure sets [Benacerraf 1965].

(ii) Formalisation of mathematics: An error-prone task

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(iii) Computer assisted proofs

- Mizar mathematical library (over 59.000 theorems from Tarski-Grothendieck set theory)
- Metamath (over 23.000 theorems from ZFC set theory)

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