Verification of Functional Programs
Preliminary Concepts

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What is a Type?

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- Types as ranges of significance of propositional functions. Let $\varphi(x)$ be a (unary) propositional function. The type of $\varphi(x)$ is the range within which x must lie if $\varphi(x)$ is to be a proposition [Russell (1903) 1938, Appendix B: The Doctrine of Types].

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- Types as ranges of significance of propositional functions. Let $\varphi(x)$ be a (unary) propositional function. The type of $\varphi(x)$ is the range within which x must lie if $\varphi(x)$ is to be a proposition [Russell (1903) 1938, Appendix B: The Doctrine of Types].

  In modern terminology, Rusell's types are domains of propositional functions.

  Example

  Let $\varphi(x)$ be the propositional function 'x is a prime number'. Then $\varphi(x)$ is a proposition only when its argument is a natural number.

$$\varphi : \mathbb{N} \to \{\mathrm{False}, \mathrm{True}\}$$
$$\varphi(x) = x \text{ is a prime number.}$$

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- ‘A type is an approximation of a dynamic behaviour that can be derived from the form of an expression.’ [Kiselyov and Shan 2008, p. 8]
- The propositions-as-types principle (Curry-Howard correspondence)
- Homotopy Type Theory (HTT)

  Propositions are types, but not all types are propositions (e.g. higher-order inductive types)

What is a Type?

Example (some Haskell's types)

- Type variables: `a`, `b`
- Type constants: `Int`, `Integer`, `Char`
- Function types: `Int → Bool`, `(Char → Int) → Integer`
- Product types: `(Int, Char)`, `(a, b)`
- Disjoint union types:

```
data Sum a b = Inl a | Inr b
```

Type Systems

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- 'A type system is a tractable syntactic method for proving the absence of certain program behaviors by classifying phrases according to the kinds of values they compute.' [Pierce 2002, p. 1]

Referential Transparency

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'A language that supports the concept that "equals can be substituted for equals" in an expression without changing the value of the expression is said to be *referentially transparent*.' [Abelson and Sussman 1996, p. 233].

Referential Transparency

Example

The following C program prints `hello, world` twice.

```
#include <stdio.h>

int
main (void)
{
  printf ("hello, world");
  printf ("hello, world");

  return 0;
}
```

Referential Transparency

Example

The following C program prints `hello, world` once.

```
#include <stdio.h>

int
main (void)
{
  int x;

  x = printf ("hello, world");
  x; x;

  return 0;
}
```

Referential Transparency

Example

The following Haskell program prints `hello, world` twice.

```haskell
main :: IO ()
main = putStr "hello, world" >> putStr "hello, world"
```

Referential Transparency

In Haskell, given

```
let x = exp
in  ... x ... x ...
```

the meaning of `... x ... x ...` is the same as `... exp ... exp ...`

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Example

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```
main :: IO ()
main = let x :: IO ()
           x = putStr "hello, world"
       in x >> x
```

Example

The following Haskell program prints `hello, world` twice.

```
main :: IO ()
main = x >> x
  where x :: IO ()
        x = putStr "hello, world"
```

Pure Functions

Side effects

'A side effect introduces a dependency between the global state of the system and the behaviour of a function... Side effects are essentially invisible inputs to, or outputs from, functions.' [O'Sullivan, Goerzen and Stewart 2008, p. 27].

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Pure functions

'Take all their input as explicit arguments, and produce all their output as explicit results.' [Hutton 2007, p. 87].

Pure Functions

Are the following GHC 7.8.2 functions, pure functions?

```
maxBound :: Int     -- Prelude
os       :: String  -- System.Info
```

*From: `https://wiki.haskell.org/Referential_transparency`, 2014-02-25.

Pure Functions

Are the following GHC 7.8.2 functions, pure functions?

```
maxBound :: Int     -- Prelude
os       :: String  -- System.Info
```

'One perspective is that Haskell is not just one language (plus `Prelude`), but a family of languages, parametrized by a collection of implementation-dependent parameters. Each such language is RT, even if the collection as a whole might not be. Some people are satisfied with situation and others are not.' *

*From: `https://wiki.haskell.org/Referential_transparency`, 2014-02-25.

Functions are First-Class Citizens

Source: Abelson and Sussman [1996]

- They can be passed as arguments and they can be returned as results (higher-order functions)
- They can be assigned to variables
- They can be stored in data structures

Bottom

Working with functions how handle undefined values yielded by partial functions or non-terminating functions?

Example

```
head :: [a] → a
head (x : _) = x

head [] = ?
```

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```haskell
head :: [a] → a
head (x : _) = x

head [] = ?
```

Example

```haskell
fst :: (a, b) → a
fst (x, _) = x

ones :: [Int]
ones = 1 : ones

fst (ones, 10) = ?
```

Bottom

The $\perp$ symbol represents the undefined value.
($\perp$ is represented in Haskell by the **`undefined`** keyword)

Example (first version)

```
head []          = undefined
fst (ones, 10) = undefined
```

*See 'Hussling Haskell types into Hasse diagrams' from Edward Z. Yang's blog on December 6, 2010.

Bottom

The $\perp$ symbol represents the undefined value.
($\perp$ is represented in Haskell by the **undefined** keyword)

Example (first version)

```
head []         = undefined
fst (ones, 10) = undefined
```

Remark

The $\perp$ value is polymorphic in Haskell.

Remark

The Haskell types are lifted types.*

*See 'Hussling Haskell types into Hasse diagrams' from Edward Z. Yang's blog on December 6, 2010.

Example (second version)

$$\text{head } [] = \bot_{\mathsf{a}}$$
$$\text{fst } (\text{ones}, 10) = \bot_{[\mathsf{Int}]}$$

Therefore, `head []` $\neq$ `fst (ones, 10)`.

Bottom

Example

```haskell
foo :: Int → Int
foo 0 = 0

bar :: Int → Int
bar n = bar (n + 1)

foobar :: Int → Int
foobar n = if foo n == 0 then 1 else 2
```

Example

```
foo :: Int → Int
foo 0 = 0

bar :: Int → Int
bar n = bar (n + 1)

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Can we replace `foo` by `bar` in `foobar`?

Bottom

Example

```
foo :: Int → Int
foo 0 = 0

bar :: Int → Int
bar n = bar (n + 1)

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foobar n = if foo n == 0 then 1 else 2
```

Can we replace `foo` by `bar` in `foobar`? Only for $n \neq 0$.

Lazy Evaluation

See slides for the chapter 12 on the book by Hutton [2007]:
http://www.cs.nott.ac.uk/~gmh/book.html.

Strict and Non-Strict Functions

Definition

Let f be a unary function. If f $\perp = \perp$ then f is a **strict** function, otherwise it is a **non-strict** function. The definition generalise to n-ary functions.

Example

The `three` function is non-strict.

```haskell
three :: a → Int
three _ = 3
```

```haskell
three undefined               = 3
three (head [])               = 3
three (fst (ones, 10))        = 3
three (putStr "hello, world") = 3
```

Strict and Non-Strict Functions

Example

```
three :: a → Int
three _ = 3
```

Non-strict reasoning...

$$(\forall x \in \mathsf{Int})(\forall y)(x + \mathsf{three}\ y = x + 3).$$

Example

(Why Haskell hasn't a predefined recursive data type for natural numbers?)

```
data Nat = Zero | Succ Nat
```

```
Zero :: Nat
Succ :: Nat → Nat
```

Strict and Non-Strict Functions

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data Nat = Zero | Succ Nat

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Is `Succ` a non-strict function?

Strict and Non-Strict Functions

Example

(Why Haskell hasn't a predefined recursive data type for natural numbers?)

```
data Nat = Zero | Succ Nat
```

```
Zero :: Nat
Succ :: Nat → Nat
```

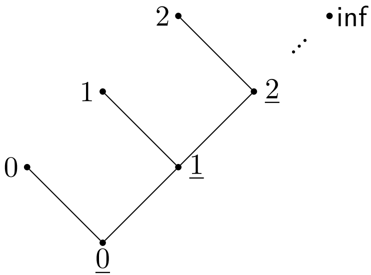
Is `Succ` a non-strict function?

We can define

```
inf :: Nat
inf = Succ inf
```

Example (cont.)

`Nat` represents the lazy natural numbers, that is, `Succ ⊥ ≠ ⊥` [Escardó 1993].



$$
\begin{aligned}
\underline{0} &= \bot, \\
\underline{n+1} &= \mathsf{Succ}\ n, \\
\mathsf{inf} &= \bigsqcup_{n \in \omega} \underline{n}
\end{aligned}
$$

Partially Ordered Sets

Definition

A **partially ordered set (poset)** $(D, \sqsubseteq)$ is a set D on which the binary relation $\sqsubseteq$ satisfies the following properties:

$$\begin{aligned} \forall x.\, x \sqsubseteq x & \qquad \text{(reflexive)} \\ \forall x\, \forall y\, \forall z.\, x \sqsubseteq y \land y \sqsubseteq z \Rightarrow x \sqsubseteq z & \qquad \text{(transitive)} \\ \forall x\, \forall y.\, x \sqsubseteq y \land y \sqsubseteq x \Rightarrow x = y & \qquad \text{(antisymmetry)} \end{aligned}$$

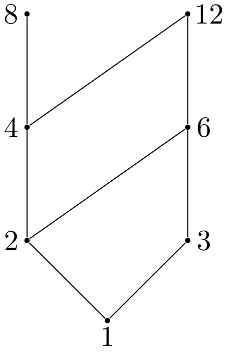
Partially Ordered Sets

Examples

- $(\mathbb{Z}, \leq)$ is a poset.
- Let $a, b \in \mathbb{Z}$ with $a \neq 0$. The divisibility relation is defined by $a \mid b := \exists c\ (ac = b)$. Then $(\mathbb{Z}^+, |)$ is a poset.
- $(P(A), \subseteq)$ is a poset.

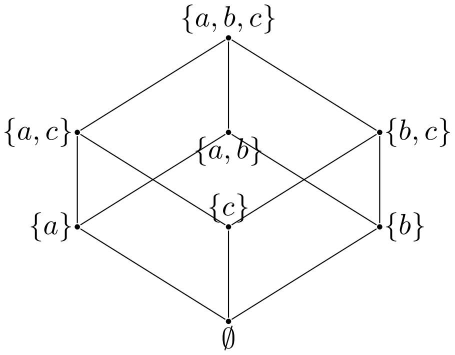
Example

Hasse diagram for the poset $(\{1, 2, 3, 4, 6, 8, 12\}, |)$.



Example

Hasse diagram for the poset $(\{a, b, c\}, \subseteq)$.



Monotone Functions

Definition

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be two posets. A function $f : D \rightarrow D'$ is **monotone** iff

$$\forall x \; \forall y. \; x \sqsubseteq y \Rightarrow f(x) \sqsubseteq' f(y).$$

Some Concepts of Fixed-Point Theory

Let D be a set, $(D, \sqsubseteq)$ be a poset and f be a function $f : D \to D$.

Definition

An element $d \in D$ is a **fixed-point** of f iff

$$f(d) = d.$$

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The **least/greatest fixed-point** of f is least/greatest among the fixed-points of f.

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Definition

The **least/greatest fixed-point** of f is least/greatest among the fixed-points of f.

That is, $d \in D$ is the least/greatest fixed-point of f iff:

- $f(d) = d$ and
- $\forall x. f(x) = x \Rightarrow d \sqsubseteq x$ / $\forall x. f(x) = x \Rightarrow x \sqsubseteq d$.

Some Concepts of Fixed-Point Theory

Theorem

Let $(D, \sqsubseteq)$ be a poset and $f : D \rightarrow D$ be monotone. Under certain conditions f has a least fixed-point [Winskel 1994] and a greatest fixed-point [Ésik 2009].

Some Concepts of Fixed-Point Theory

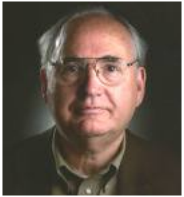
Theorem

Let $(D, \sqsubseteq)$ be a poset and $f : D \to D$ be monotone. Under certain conditions f has a least fixed-point [Winskel 1994] and a greatest fixed-point [Ésik 2009].

Notation

The least and greatest fixed-points of f are denoted by $\mu x.f(x)$ and $\nu x.f(x)$, respectively.

Motivation: Does λ-calculus have models?



'Historically my first model for the λ-calculus was discovered in 1969 and details were provided in Scott [1972] (written in 1971).' [Scott 1980, p. 226.].

Introduction to Domain Theory

Non-standard definitions

pre-domain, domain, complete partial order (cpo), ω-cpo, bottomless ω-cpo, Scott's domain, ...

Convention

domain $\equiv$ ω-complete partial order

ω-Complete Partial Orders

Definition

Let $(D, \sqsubseteq)$ be a poset. A ω-**chain** of D is an increasing chain

$$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$$

where $d_i \in D$.

ω-Complete Partial Orders

Definition

Let $(D, \sqsubseteq)$ be a poset. The poset D is a ω-**complete partial order** (ω-cpo) iff [Plotkin 1992]:

1. There is a least element $\bot \in D$, that is, $\forall x. \bot \sqsubseteq x$. The element $\bot$ is called *bottom*.
2. For every increasing ω-chain $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$, the least upper bound $\bigsqcup_{n \in \omega} d_n \in D$ exists.

ω-Complete Partial Orders

Definition

Let A be a set. The symbol $A_\perp$ denotes the ω-cpo whose elements $A \cup \{\perp\}$ are ordered by

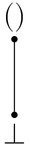
$$x \sqsubseteq y \quad \text{iff} \quad x = \perp \text{ or } x = y.$$

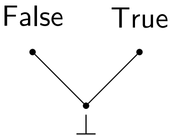
The ω-cpo $A_\perp$ is called A **lifted** [Mitchell 1996].

ω-Complete Partial Orders

Examples

The lifted unit type and the lifted Booleans $B_\perp$ are ω-cpos.





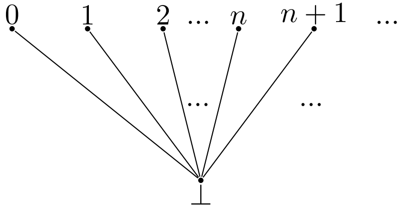

```
data () = ()
```

```
data Bool = True | False
```

ω-Complete Partial Orders

Example

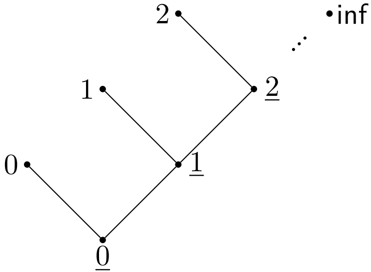
The lifted natural numbers $N_\perp$.



Example

The lazy natural numbers ω-cpo.

```haskell
data Nat = Zero | Succ Nat
```



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\begin{aligned}
\underline{0} &= \bot, \\
\underline{n+1} &= \mathsf{Succ}\ n, \\
\mathsf{inf} &= \bigsqcup_{n\in\omega} \underline{n}
\end{aligned}
$$

$$\bigsqcup_{n\in\omega} \underline{n} = \bot \sqsubseteq \mathsf{Succ}\ \bot \sqsubseteq \mathsf{Succ}\ (\mathsf{Succ}\ \bot) \sqsubseteq \cdots$$

Admissible Properties

Definition

Let D be a w-cpo. A property P (a subset of D) is w-**inductive (admissible)** iff whenever $\langle x_n \rangle_{n \in \omega}$ is an increasing sequence of elements in P, then $\bigsqcup_{n \in \omega} x_n$ is also in P, that is,

$$\forall n \in \omega.\, P(x_n) \Rightarrow P\left(\bigsqcup_{n \in \omega} x_n\right).$$

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