Verification of Functional Programs
Induction

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Source Code

All the source code have been tested with Agda 2.3.2, Coq 8.4pl3 and Isabelle 2013-2.

The Principle of Mathematical Induction

The principle of mathematical induction

Let $A(x)$ be a propositional function. To prove $A(x)$ for all $x \in \mathbb{N}$, it suffices prove:

- the basis $A(0)$ and
- the induction step, that $A(n) \Rightarrow A(n+1)$, for all $n \in \mathbb{N}$
  ($A(n)$ is called the induction hypothesis).

The Principle of Mathematical Induction

First-order logic version

Let $A(x)$ be a formula with free variable x. For each formula $A(x)$:

$$[\,A(0) \wedge \forall x.A(x) \Rightarrow A(x+1)\,] \Rightarrow \forall x.A(x) \quad \text{(axiom schema of induction)}$$

The Principle of Mathematical Induction

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Equivalent formulations

$$A(0) \Rightarrow [\, (\forall x. A(x) \Rightarrow A(x+1)) \Rightarrow \forall x. A(x) \,] \qquad \text{(by exportation)}$$

$$A(0) \Rightarrow (\forall x. A(x) \Rightarrow A(x+1)) \Rightarrow \forall x. A(x) \qquad \text{(right-assoc. conditional)}$$

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Inference rule style

$$\frac{A(0) \qquad \forall x.A(x) \Rightarrow A(x+1)}{\forall x.A(x)}$$

The Principle of Mathematical Induction

Higher-order logic

'The adjetive 'first-order' is used to distinguish the languages... from those in which are predicates having other predicates or functions as arguments, or quantification over functions or predicates, or both.' [Mendelson (1965) 1997, p. 56]

The Principle of Mathematical Induction

Higher-order logic

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Second-order logic version

Let X be a predicate variable.

$$\forall X.X(0) \Rightarrow (\forall x.X(x) \Rightarrow X(x+1)) \Rightarrow \forall x.X(x) \qquad \text{(axiom of induction)}$$

The Principle of Mathematical Induction

Historical remark

Dedekind [(1888) 2005] and Peano [(1889) 1967] axiom: $1 \in \mathbb{N}$.

Remark

Coq generates the induction principles associated to the inductively defined (data) types.

Example (Coq)

The inductive data type for natural numbers.

```
Require Import Unicode.Utf8.

Inductive nat : Set :=
| O : nat
| S : nat → nat.
```

Continued on next slide

Example (continuation)

The `Check nat_ind` command yields:

```
nat_ind : ∀ P : nat → Prop,
          P 0 → (∀ n : nat, P n → P (S n)) → ∀ n : nat, P n
```

The Principle of Mathematical Induction

Example (continuation)

The `Check nat_ind` command yields:

```
nat_ind : ∀ P : nat → Prop,
          P 0 → (∀ n : nat, P n → P (S n)) → ∀ n : nat, P n
```

The `Check nat_rec` command yields:

```
nat_rec : ∀ P : nat → Set,
          P 0 → (∀ n : nat, P n → P (S n)) → ∀ n : nat, P n
```

Example (continuation)

The `Check nat_ind` command yields:

```
nat_ind : ∀ P : nat → Prop,
            P 0 → (∀ n : nat, P n → P (S n)) → ∀ n : nat, P n
```

The `Check nat_rec` command yields:

```
nat_rec : ∀ P : nat → Set,
            P 0 → (∀ n : nat, P n → P (S n)) → ∀ n : nat, P n
```

The `Check nat_rect` command yields:

```
nat_rec : ∀ P : nat → Type,
            P 0 → (∀ n : nat, P n → P (S n)) → ∀ n : nat, P n
```

The Principle of Mathematical Induction

Implementation remark

What happen if instead of using

```
Inductive nat : Set := 0 : nat | S : nat → nat
```

we renamed the data type nat by

```
Inductive P : Set := 0 : P | S : P → P
```

or we renamed the data constructor S by

```
Inductive nat : Set := 0 : nat | P : nat → nat
```

?

Source: McBride and McKinna [2004]

The Principle of Mathematical Induction

Remark

Isabelle also generates the induction principles associated to the inductively defined (data) types.

Example (Isabelle)

The inductive data type for natural numbers.

```
datatype nat = Z | S nat
```

The Principle of Mathematical Induction

Remark

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Example (Isabelle)

The inductive data type for natural numbers.

```
datatype nat = Z | S nat
```

The `print_theorems` command yields (among others):

```
nat.induct: ?P Z ⇒ ∀x. ?P x ⇒ ?P (S x)) ⇒ ?P ?nat
```

Remark

Agda doesn't generate the induction principles, but the user can use pattern matching on the inductively defined (data) types.

Example (Agda)

The inductive data type for natural numbers.

```
data ℕ : Set where
  zero : ℕ
  succ : ℕ → ℕ
```

Continued on next slide

Example (continuation)

The principle of mathematical induction.

```
ℕ-ind : (A : ℕ → Set) →
        A zero →
        (∀ n → A n → A (succ n)) →
        ∀ n → A n
ℕ-ind A A0 h zero     = A0
ℕ-ind A A0 h (succ n) = h n (ℕ-ind A A0 h n)
```

The Principle of Mathematical Induction

Remark

In Agda, Coq and Isabelle, the ‘axiom of induction’ is not an axiom

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In Agda, Coq and Isabelle, the ‘axiom of induction’ is not an axiom (the introduction rules induce the induction principles).

Course-of-Values Induction

Course-of-values induction (strong or complete induction)

Let $A(x)$ be a propositional function. To prove $A(x)$ for all $x \in \mathbb{N}$, it is enough to prove:

$$(\forall 0 \leq k < n)(A(k) \Rightarrow A(n)), \text{ for all } n \in \mathbb{N}.$$

Example

The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$ and $F_{k+2} = F_k + F_{k+1}$, so $F = \{0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$.

Course-of-Values Induction

Example

The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$ and $F_{k+2} = F_k + F_{k+1}$, so $F = \{0, 1, 1, 2, 3, 5, 8, 13, 21, \dots\}$.

Let Φ and $\hat{\Phi}$ be the roots of the equation $x^2 - x - 1$:

$$\Phi = \frac{1+\sqrt{5}}{2} \text{ and } \hat{\Phi} = \frac{1-\sqrt{5}}{2},$$

so $\Phi^2 = \Phi + 1$ and $\hat{\Phi}^2 = \hat{\Phi} + 1$. Then [Bird and Wadler 1988, p. 107.]

$$F_k = \frac{1}{\sqrt{5}}(\Phi^k - \hat{\Phi}^k), \text{ for all } k \in \mathbb{N}.$$

Mathematical and Course-of-Values Induction

Theorem

Mathematical induction and course-of-values induction are equivalent [Winskel 2010].

Structural induction

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To prove $A(X)$ for all the structures X, it suffices prove [Hopcroft, Motwani and Ullman 2007]:

- $A(X)$ for the basis structure(s) of X and

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To prove $A(X)$ for all the structures X, it suffices prove [Hopcroft, Motwani and Ullman 2007]:

- $A(X)$ for the basis structure(s) of X and
- given a structure X whose recursive/inductive definition says is formed from $Y_1, \dots, Y_k$, that $A(X)$ assuming that the properties $A(Y_1), \dots, A(Y_k)$ hold.

Structural Induction for Lists

Example (Coq)

The parametric inductive data type.

```
Require Import Unicode.Utf8.

Inductive list (A : Type) : Type :=
| nil  : list A
| cons : A → list A → list A.
```

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| nil  : list A
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```

The induction principle.

```
list_ind : ∀ (A : Type) (P : list A → Prop),
           P (nil A) →
           (∀ (a : A) (l : list A), P l → P (cons A a l)) →
           ∀ l : list A, P l
```

Structural Induction for Lists

Example (Isabelle)

The polymorphic inductive data type.

```
datatype 'a list = Nil | Cons 'a "'a list"
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The induction principle.

```
list.induct: ?P Nil ⇒ ∀x1 x2. ?P x2 ⇒ ?P (Cons x1 x2)) ⇒
             ?P ?list
```

Structural Induction for Lists

Example (Agda)

The parametric inductive data type.

```
data List (A : Set) : Set where
  []  : List A
  _∷_ : A → List A → List A
```

Structural Induction for Lists

Example (Agda)

The parametric inductive data type.

```
data List (A : Set) : Set where
  []  : List A
  _∷_ : A → List A → List A
```

The induction principle.

```
List-ind : {A : Set} (B : List A → Set) →
           B [] →
           ((x : A) (xs : List A) → B xs → B (x ∷ xs)) →
           ∀ xs → B xs
List-ind B B[] h []       = B[]
List-ind B B[] h (x ∷ xs) = h x xs (List-ind B B[] h xs)
```

Well-Founded Induction

Definition

Let $\prec$ be a binary relation on a set A. The relation $\prec$ is a **well-founded** relation iff every non-empty subset $S \subseteq A$ has a minimal element, that is,

$$(\forall S \subseteq A)[\, S \neq \emptyset \Rightarrow (\exists m \in S)(\forall s \in S)(s \nprec m)\,].$$

Well-Founded Induction

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$$(\forall S \subseteq A)[\, S \neq \emptyset \Rightarrow (\exists m \in S)(\forall s \in S)(s \not\prec m)\,].$$

Definition (Well-founded induction)

Let $\prec$ be a well-founded relation on a set A and $A(x)$ a propositional function. To prove $A(x)$ for all $a \in A$, it suffices prove:

$$(\forall b \prec a)(A(b) \Rightarrow A(a)), \text{ for all } a \in A.$$

Well-Founded Induction

Example

Let $\prec$ be the well-founded relation on $\mathbb{N}$ given by the graph of the successor function $n \mapsto n+1$.

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Example

Let $\prec$ be the well-founded relation 'less than' on $\mathbb{N}$.

Then course-of-values induction is a special case of well-founded induction.

Well-Founded Induction

Example

'If we take $\prec$ to be the relation between expressions such that $a \prec b$ holds iff a is an immediate sub-expression of b we obtain the principle of structural induction as a special case of well-founded induction.' [Winskel 2010, p. 93]

Empty Type

In type theory $a : A$ denotes that a is a term (or proof term) of type A.

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Under the proposition-as-types principle, the empty type represents the false (absurdity or contradiction) proposition [Sørensen and Urzyczyn 2006].

Therefore `e : EmptyType` represents a contradiction in our formalisation.

Empty Type

Example (Agda)

```
data ⊥ : Set where

⊥-elim : {A : Set} → ⊥ → A
⊥-elim ()  -- The absurd pattern.
```

Empty Type

Example (Coq)

(From the standard library)

```
Inductive Empty_set : Set :=.

Empty_set_rect : ∀ (P : Empty_set → Type) (e : Empty_set), P e
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Inductive Empty_set : Set :=.

Empty_set_rect : ∀ (P : Empty_set → Type) (e : Empty_set), P e

Theorem emptySetElim {A : Set}(e : Empty_set) : A.
  apply (Empty_set_rect (fun _ => A) e).
Qed.
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Inductive Empty_set : Set :=.

Empty_set_rect : ∀ (P : Empty_set → Type) (e : Empty_set), P e

Theorem emptySetElim {A : Set}(e : Empty_set) : A.
  apply (Empty_set_rect (fun _ => A) e).
Qed.

Theorem emptySetElim' {A : Set}(e : Empty_set) : A.
  elim e.
Qed.
```

Remark

The inductive types can be defined/represented as least fixed-points of appropriated functions (functors).

Strictly Positive Inductive Types

Remark

The inductive types can be defined/represented as least fixed-points of appropriated functions (functors).

Example

Let 1 be the unity type, and $+$ and $\times$ be the operators for disjoint union and Cartesian product, respectively. Then

$$\begin{aligned} \mathsf{Nat} &:= \mu X.1 + X, \\ \mathsf{List}\ A &:= \mu X.1 + (A \times X). \end{aligned}$$

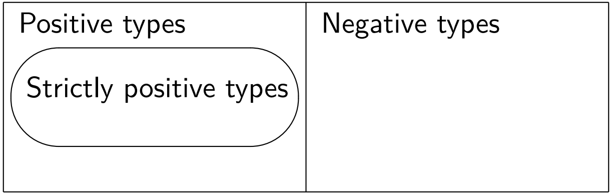
Strictly Positive Inductive Types

Definition

'The occurrence of a type variable is **positive** iff it occurs within an even number of left hand sides of $\rightarrow$-types, it is **strictly positive** iff it never occurs on the left hand side of a $\rightarrow$-type.' [Abel and Altenkirch 2000, p. 21].

Definition

Let $\mu X.F(X)$ be an inductive type. The type $\mu X.F(X)$ is a **strictly positive type** if X occurs strictly positive in $F(X)$.



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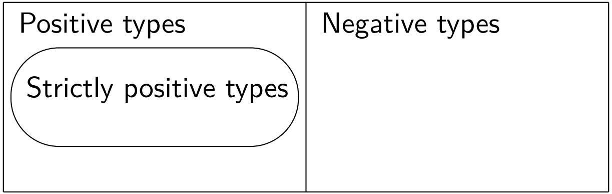

Proof assistants

Agda, Coq and Isabelle accept only strictly positive inductive types.

Strictly Positive Inductive Types

Some issues with non-strictly positive inductive types

- Infinite unfolding
  See source code in the course web page.

Strictly Positive Inductive Types

Some issues with non-strictly positive inductive types

- Infinite unfolding
  See source code in the course web page.
- Proving absurdity
  See source code in the course web page.

Strictly Positive Inductive Types

The following examples of inductive types* are rejected by Agda (Coq and Isabelle) because they are not strictly positive inductive types.

Example (negative type)

$$\mathsf{D} := \mu X.X \to X$$

```
data D : Set where
  lam : (D → D) → D
-- D is not strictly positive, because it occurs to the left
-- of an arrow in the type of the constructor lam in the
-- definition of D.
```

*Adapted from the Coq'Art, Matthes' PhD thesis and Agda's source code.

Strictly Positive Inductive Types

Example (positive, non-strictly positive type)

$$\mathsf{P} := \mu X.(X \to 2) \to 2$$

```
data P : Set where
   p : ((P → Bool) → Bool) → P
-- P is not strictly positive, because it occurs to the left
-- of an arrow in the type of the constructor p in the
-- definition of P.
```

References

Abel, Andreas and Altenkirch, Thorsten (2000). A Predicative Strong Normalisation Proof for a λ-Calculus with Interleaving Inductive Types. In: Types for Proofs and Programs (TYPES 1999). Ed. by Coquand, Thierry et al. Vol. 1956. Lecture Notes in Computer Science. Springer, pp. 21–40 (cit. on p. 50).

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Sørensen, Morten-Heine and Urzyczyn, Paul (2006). Lectures on the Curry-Howard Isomorphism. Vol. 149. Studies in Logic and the Foundations of Mathematics. Elsevier (cit. on pp. 41–43).

Winskel, Glynn (2010). Set Theory for Computer Science. (Cit. on pp. 24, 40).