PROPOSAL REPORT

Computational Implementation of the Calculation of some Integrals related to the Wavelet-Galerkin Method

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1 PROBLEM DESCRIPTION

In the application of the Galerkin-Wavelet method on the seek of solving different differential problems it turns out that it is necessary to evaluate a series of some integrals whose analytical solution has not been found and whose numerical one requires the use of complex algorithms in order to approximate them. These integral, known as connection coefficients, are a result of the construction of the *Daubechies* wavelets [1], whose properties allow to build different approximations of derivatives and non-linear terms concerning the Galerkin-Wavelet scheme on ordinary or partial differential equations.

Let L be a differential operator defined on a dense set $\mathscr{D}_L$ over the *Hilbert* space $L^2([0,1])$ and f a given function. The problem is to find an approximate solution of the equation

$$\begin{aligned} &Lu = f \quad \text{on} \quad \mathscr{D}_L, \\ &\text{with Dirichlet boundary conditions} \quad u(0) = a, u(1) = b \end{aligned} \tag{1}$$

The Galerkin technique consists of considering ϕ_i as a base of $L^2([0,1])$ and every ϕ_i satisfying C^2 on $[0,1]$ such that $\phi_i(0) = a, \phi_i(1) = b$, u_0 with Λ as a finite set of indices i and S the subspace $span\{\phi_i : i \in \Lambda\}$ so that

$$\langle Lu_0 - f, \phi_i \rangle = 0, \quad \forall i \in \Lambda \tag{2}$$

then for it follows that $Lu_0 - f = 0$ in $L^2([0,1])$ and therefore u_0 is a solution for the problem (1). Letting $\tilde{u}$ be the approximate solution of the given problem of the form (3) such that (2) is satisfied, that is to make the residue $R = L\tilde{u} - f$ to be orthogonal to the chosen base on $\mathscr{D}_L$.

$$\tilde{u} = \sum_{k \in \Lambda} a_k \phi_k \tag{3}$$

The Galerkin-Wavelet method takes $\Psi_{j,k}(x) = \phi(x) = 2^{j/2}\Psi(2^j x - k)$ as a wavelet basis for $L^2([0,1])$ satisfying the boundary conditions $\Psi_{j,k}(0) = \Psi_{j,k}(1) = 0$ and $\forall j,k \in \Lambda$ then $\Psi_{j,k}$ is C^2. Replacing the last expression into (3) then

$$\tilde{u} = \sum_{j,k \in \Lambda} a_{j,k} \Psi_{j,k}.$$

Notice that (2) can also be written as

$$\langle L\tilde{u}, \phi_i \rangle = \langle f, \phi_i, \quad \forall i \in \Lambda, \quad \text{substituing } \tilde{u}$$
$$\sum_{j,k\in\Lambda} \langle L\Psi_{j,k}, \Psi_{l,m} \rangle a_{j,k} = \langle f, \Psi_{l,m} \rangle, \quad \forall l, m \in \Lambda$$

As commented before, the *Daubechies* wavelets play a fundamental role in the construction of $a_{j,k}$ and $\Psi_{j,k}$ since they form a compactly supported orthonormal base which includes members from highly localized to highly smooth. Such wavelets are determined by a set of L coefficients (genus of *Daubechies* wavelet) $\{p_k : k = 0, 1, \ldots, L-1\}$, p_k denoted as wavelet filter coefficients, through the relation

$$\varphi(x) = \sum_{k=0}^{L-1} a_k \varphi(2x - k)$$

and the equation

$$\Psi(x) = \sum_{k=2-L}^{L} (-1)^k p_{1-k} \varphi(2x - k),$$

where $\varphi(x)$ and $\Psi(x)$ are called scalling function and mother wavelet with supports are in the interval $[0, L-1]$ and $[1 - \frac{L}{2}, \frac{L}{2}]$ respectively.

In the development of the method it is necessary the construction of orthogonal wavelets, which could be interval wavelets or defined on the whole space, and procedures to evaluate the integrals in (4) related to this process.

$$\begin{aligned}
\varphi^{(n)}(x) &= \frac{\mathrm{d}^n \varphi(x)}{\mathrm{d}x^n}, \\
\theta_n(x) &= \underbrace{\int_0^x \int_0^{y_n} \cdots \int_0^{y_2}}_{n-tuple} \varphi(y_1)\mathrm{d}y_1 \cdots \mathrm{d}y_{n-1}\mathrm{d}y_n, \\
M_k^m(x) &= \int_0^x y^m \varphi(y-k)\mathrm{d}y, \\
\Gamma_k^n(x) &= \int_0^x \varphi^{(n)}(y-k)\varphi(y)\mathrm{d}y, \\
\Lambda_k^{m,n}(x) &= \int_0^x y^m \varphi^{(n)}(y-k)\varphi(y)\mathrm{d}y, \\
\Upsilon_k^{m,n}(x) &= \int_0^x y^m \theta_n(y-k)\varphi(y)\mathrm{d}y, \\
\Omega_{j,k}^{m,n}(x) &= \int_0^x \varphi(y)\varphi^{(m)}(y-j)\varphi^{(n)}(y-k)\mathrm{d}y.
\end{aligned} \tag{4}$$

Usually, when dealing with bounded interval the previous functionals must be computated, where $j,k \in \mathbb{Z}$, $m,n,x \in \mathbb{Z}^+ + \{0\}$, and $\mathbb{Z}$ and $\mathbb{Z}^+$ denote the sets of integers and positive integers, respectively, otherwise, when solving for unbounded intervals then x must be taken at the limit, i.e, $x \to \infty$.

2 OBJECTIVES

2.1 General objective

The main objective of this project is to perform a computational implementation of the Wavelet-Galerkin method through the discretized form of several integrals related to the method. Having done this, it will be possible to verify several results coming from the literature. In this way, since the project has already been worked by the advisor's research group, such implementation will be displayed in a practical way so the main practical results can be kept for numerical posterior analysis and observation.

2.2 Specific objectives

- Study the Wavelet-Galerkin method theory for the solution of differential equations.
- Apply the Wavelet-Galerkin method in order to solve ordinary differential equations, as a first step to discuss more complex and difficult problems.
- Implement the Wavelet-Galerkin method to approximate the solution of some differential equations, whose exact solution is already known.

3 JUSTIFICATION

The development of methods able to result into approximations of differential equations, ordinary and partial, is certainly a subject of general interest in the industry and academical society since usually they describe processes underlying real phenomena whose mathematical description and simulation are generally use to increase its profitableness or to understand its behavior; nevertheless, these approximations are usually seek to satisfy certain conditions related to the good behavior of the solution as the time increases or to the initial and boundary conditions imposed. Therefore, these methods must be improved or new others are introduced so that a equilibrium between performance accuracy and computational resources can be reached.

The wavelets are a mathematical object whose properties, such as orthogonality, compact support, good representation of polynomials and ability to represent functions at different levels of resolution [2] are desirable and useful when dealing with basis for a Galerkin approach to solve differential equations. The study of techniques based on wavelets has let the introduction of a new family of numerical methods whose computational requirements are considerably less than those by finite differences and finite elements; nonetheless, the implementation of such methods is not arbitrary and depend upon whether it is computable or not and of the quality of the discretized scheme. By the exploitation of the properties of the wavelet basis functions one wants to find simpler ways to solve the integral operations associated to the method [3] coming from the need of determinate the values of inner products required to evaluate the method results.

4 Precedent Research

The wavelet yield to the construction of mathematical models similar to those coming from the variational calculus, which deals with with maximization or minimization of functionals, and have generated a great interest due to its properties. On one side, the theorical research on this subject has increased widely since the introduction of the *Haar* system [4], due to the Hungarian mathematician *Alfred Haar* around 1910, which allowed to represent a target function over an interval in terms of an orthogonal function basis, but it had the disadvantage of not being continuous and thefore not differential, both essential conditions to be satisfied in further and more complex analysis. Later in 1988, the Belgian physicist *Ingrid Daubechies* proposed the construction of orthogonal wavelets with compact support and proves some of their most relevant properties, including the possibility of compute their connection coefficients. In terms of the *Galerkin* method, wavelet basis functions are considerably practical as a result of their characteristics and therefore the test function is considered as a linear combination of a wavelet basis.

On the other hand, as a mathematical tool, wavelets can be used to extract information from many different kinds of data, also to compress it and are also used to represent a wide variety of curves and surfaces in CADD (computer-aided design and drafting) software, hence are of great interest in applied mathematics and engineering [5]. Thus, in terms of practical applications and theorical use potential the improvement and exploration of this method,

specially as the element capable of produce enhancement on the numerical methods to approximate the solution of differential equations, is of high significance.
When applying this last method for solving ordinary or partial differential equations some improper connection coefficients result as terms in its equations but if it is related to a bounded interval produces proper connection coefficients [3]. Either way, in order to compute these coefficients several techniques have been proposed, which take advantage of the nature of the basis functions to build series approximations for later calculations via algorithms for the functionals (4). In a first stage, *Latto et al.* [6], give a first approach to the calculation of these coefficients by imposing periodic boundary conditions and deriving a formula to compute the moments by induction but he provides connection coefficients only for two cases. Later, *Mishra* [7], *Lin* [8], *Popovici* [9] and others worked on the same subject and were able to extend the computations for all the cases under different techniques.

5 SCOPE

Althought we know there are already several academical works on the same subject is our intention to be able to validate, in a first stage, their results using the algorithms in the respective papers. Then, having understood the most relevant theory related to this subject then, if possible, propose different alternatives for the same calculations. Thefore, our scope would be to enforce the results and conclussions already present in the literature and later improve the methods, algorithms and classical techniques employed to derive approximations of several differential equations. In case we get the expected results in a short lapse of time then we would extend the methodology to more complex initial value problems and in a final stage work with simple partial differential equations, althought the quantity of computational resources we take at the begining will give us a first sight of what we could expect for more complete cases.

6 METHODOLOGY

In order to achive the objective of this work we will employ minimum five hours per week which would be distributed in short sessions of two hours with the tutor and individual research. In such meeting both student and tutor would discuss the progress on the achivement of the objectives mentioned before as well questions arising from the provided theory or the incoming one. The student will expose, as many times as required, his theorical and practical learning and achievements to his tutor so that a proper tracing of the project can be made.

7 RESOURCES

One one hand, since we expect, according to our scope, to focus our work on the computational aspect of the problem and therefore we will be using a good part of the time the computers and softwares provided by the University. On the other hand, we will be using the bibliographical databases to access to different academical articles online. Having finished the construction of the discretized problem then licensed softwares like MATLAB® [10] will be essential to continue with the development of the project.

8 INTELLECTUAL PROPERTY

According to the EAFIT's University internal intellectual property reglamentation [11], the patrimonial rights over the academic products resulting from this project will belong to the authors, Obed Ríos-Ruiz and Patricia Gómez-Palacio in a *30%*, to the *Functional Analisys Research Group* in a *15%* and to the University in a *55%*.

9 ACTIVITY SCHEDULE

We propose the following schedule letting the student use a good amount of time to develop, together and closely with his tutor, a good understanding on the activity in order to achieve the project specific objectives and therefore the general one.

Activities schedulling		
Activity	**Description**	**Week range**
Proposal report		3 - 4
Proposal presentation		5
Literature review	Search on bibliographical databases	3 - 6
Study of the appropriate literature found	Selection and collection of the most relevant literature	6 - 7
Theorical model construction	Definition of the problem variational aspect, formulation of the theorical and numerical scheme	7 - 10
Oral progress report		10
Algorithms contruction	Using the theorical basis build a secuence through which the computations can be made	10 - 13
Computational implementation of the algorithms	Using a mathematical software, such as MATLAB® [10], formulate a routine whose outputs are the connection coefficients	13 - 16
Project report		16
Project presentation		19

BIBLIOGRAPHY

[1] I. Daubechies, "Orthonormal bases of compactly supported wavelets," *Communications on Pure and Applied Mathematics*, vol. 41, no. 7, pp. 909–996, 1988.

[2] M.-Q. Chen, C. Hwang, and T.-P. Shih, "The computation of Wavelet-Galerkin approximation on a bounded interval," *International Journal for Numerical Methods in Engineering*, vol. 39, pp. 2921–2944, 1996.

[3] C. Romine and B. Peyton, "Computing connection coefficients of compactly supported wavelets on bounded intervals," *Mathematical Sciences Section, Oak Ridge National Laboratory*, 1997.

[4] A. Haar, "Zur theorie der orthogonalen funktionensysteme," *Mathematische Annalen*, vol. 71, no. 1, pp. 38–53.

[5] L. R. and S. M. Castro, "Wavelets y sus aplicaciones," in *1er Congreso Argentino de Ciencias de la Computación*, pp. 195–2014, 1995.

[6] A. Latto, H. Resnikoff, and E. Tenenbaum, "The evaluation of connection coefficients of compactly supported wavelets," *Proccedings of the French-USA Wokshop on wavelets and turbulence*, 1991.

[7] V. Mishra and Sabina, "Wavelet-Galerkin solutions of ordinary differential equations," *International Journal of Mathematics*, vol. 5, no. 9, pp. 407–424, 2011.

[8] E. Lin and X. Zhou, "Connection coefficients on an interval and wavelet solutions of Burgers equation," *Journal of Computational and Applied Mathematics*, vol. 135, no. 1, pp. 63 – 78, 2001.

[9] C. I. Popovici, "Matlab evaluation of the $\omega_{j,k}^{m,n}(x)$ coefficients for partial differential equations solving by Wavelet-Galerkin approximation," *Analele Ştiinţifice ale Universităţii "Ovidius" Constanţa. Seria: Matematică*, vol. 18, pp. 287–294, 2010.

[10] MATLAB, *version 8.5.0 (R2015a)*. Natick, Massachusetts: The MathWorks Inc., 2015.

[11] Universidad EAFIT, *Reglamento de Propiedad Intelectual*. Universidad EAFIT, 2009.