Modeling Colombian yields with a macro-factor affine term structure model

Research practise 3: Project proposal

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I. Problem formulation

Interest rates vary over time and maturity (investment time horizon). The (time varying) relationship between interest rates and their maturities is commonly called the 'term structure of interest rates' (TS for short). The TS is partially observable through the yields at which different bonds of the same issuer are traded in the market.

Modeling and forecasting the TS is useful for pricing financial instruments, managing risk and informing monetary policy. Affine term structure models (ATSMs) model the TS as an affine function[1] of a state vector. The state vector changes over time (t) and the affine function varies with maturity (τ), which gives ATSMs the required versatility to model yield curves.

We will part from a three-factor Gaussian ATSM, which is set up as follows.

The state vector $X(t)$ is assumed to follow an affine diffusion process under the risk-neutral probability measure Q:

$$dX(t) = \mu^Q(X)dt + dW^Q(t)$$

where $\mu^Q : \mathbb{R}^3 \to \mathbb{R}^3$ is an affine function and W^Q is a 3-dimensional independent brownian motion under Q.

We denote the continuously compounded yield with maturity τ at time t by $\gamma_\tau(t)$. The short rate $r(t)$ is also assumed to be affine on $X(t)$:

$$r(t) = \lim_{\tau \to 0} \gamma_\tau(t) = \delta_0 + \delta_1^\top X(t)$$

where $\delta_0 \in \mathbb{R}$ and $\delta_1 \in \mathbb{R}^3$.

If the no arbitrage hypothesis holds, the price of a pure discount bond with maturity τ at time t should be given by the following equation:

$$P(X(t),\tau) = E^Q\left[exp\left(-\int_t^{t+\tau} r(u)\,du\right)|X(t)\right]$$

where E^Q denotes the conditional expected value under Q.

Duffie and Kan (1996) show that for these assumptions to hold, $\gamma_\tau(t)$ must also be an affine functions of the state vector for every τ:

$$\gamma_\tau(t) = A(\tau) + B(\tau)^\top X(t) \quad (1)$$

where $A(\tau)$ and $B(\tau)$ are obtained from a set of ordinary differential equations that result from the imposition of no-arbitrage.

To obtain the state dynamics under the physical probability measure P, the market price of risk $\Lambda(X(t)) : \mathbb{R}^3 \to \mathbb{R}^3$ must be defined. Depending on its specification, the state dynamics might or might not be affine under P:

$$dX(t) = (\mu^Q(X) + \Lambda(X))dt + dW^P(t)$$

[1] A function $F : \mathbb{R}^N \to \mathbb{R}^M$ is said to be affine if $F(X) = A + B * X$ for some vector A and matrix B.

In a previous work (Velásquez-Giraldo and Restrepo-Tobón (2016)) we tested the performance of various ATSMs in fitting and forecasting daily Colombian yields. The main findings of that study were:

- Out of the tested models, a three-factor Gaussian ATSM performed best.
- The estimation procedure (closed-form likelihood expansions) is not adequate for our purposes and produces very complex optimization problems.
- Short maturity yields are harder to forecast using latent factors only. The inclusion of macroeconomic models might improve these results.

In this research practise we will attempt to solve the aforementioned issues and test various ways of including macroeconomic variables in ATSMs using Colombian data. We will focus on a Gaussian three-factor model, which allows for more versatility in estimation, as its transition densities are known. Beginning with the standard form presented in this Section, we will modify the model to include various macroeconomic variables and test whether these modifications affect the model's performance in fitting and forecasting Colombian interest rates.

We consider this project to be pertinent for a research practice not only as a continuation of the developments achieved the student's 'Research practise 2' course, but also as an application of the knowledge that he has acquired in recent courses. The model formulation, enhancement, and estimation will use concepts and methodologies from stochastic processes, experimental modeling, and optimization. The emphasis courses that the student has taken will also help in the interpretation and inclusion of macroeconomic variables.

II. Objectives

A. General objective

To propose, estimate, and test a Gaussian ATSM that incorporates macroeconomic variables to model and forecast the Colombian yield curve.

B. Specific objectives

The following objectives must be achieved in order to accomplish our general goal:

- Select and implement an estimation methodology for Gaussian ATSMs.
- Analyze and select possible macroeconomic variables that could be included in the model.
- Modify the baseline model to include macroeconomic variables and test its estimation and fit.
- Use the modified models to forecast Colombian interest rates and compare their performance with the results from Velásquez-Giraldo and Restrepo-Tobón (2016).

III. Literature review

ATSMs were first proposed by Duffie and Kan (1996) as a way of modeling the TS while keeping consistency with financial theory. Their general specification encompassed the two one-factor models from Vasicek (1977) and Cox et al. (1985), which are still widely used. However, the attempts of closely matching stylized facts of the yield curve quickly pushed researchers to use multi-factor models with different volatility structures and specifications of the risk price.

The most popular classification of ATSMs was provided by Dai and Singleton (2000) who segregated models by their number of factors and how may of them affect conditional volatility. They also proposed a canonical representation for ATSMs, outlining admissibility restrictions for their parameters. Properties and capabilities of each 'family' of models were discussed.

Estimation of ATSMs became increasingly challenging as model complexity grew. The transition density functions of the more complex models are not know in closed form, which has filled the literature with fixes and approximations (as Ait-Sahalia and Kimmel (2010) and Duan and Simonato (1999)) over which consensus hasn't been achieved. Another problematic aspect of the models is the optimization in their estimation procedures: objective function evaluations are expensive, the number of parameters is high, and restrictions are complex at times.

Kalman-filtering has been a popular methodology of estimation as it generates optimal parameter estimates for models in which transition densities are Gaussian. Duan and Simonato (1999) used this approach and also claimed that the Kalman filter yields reasonable estimators even for non-Gaussian models. Brandt and He (2002) argue that Kalman-filtering is inadequate for non-Gaussian multi-factor models and present a simulation-based correction that reduces the skewness and variability of the estimated parameters, but is computationally intensive.

Another important branch of the estimation methodologies has been guided by the assumption that the state can be (indirectly) observed without error. Duffee (2002), for instance, assumes that three yields of fixed maturities are observed without error: as he works with three-factor models, Equation 1 can be inverted to obtain the state time series. Parameters are found by maximizing the likelihood of the (known) state under a given model and the likelihood of observation errors (of other maturities) under the assumption that they are Gaussian.

This approach still suffers from state transition densities not being available in closed form for most models. Ait-Sahalia and Kimmel (2010) used the closed form approximations for transition densities derived in Ait-Sahalia (2008) to estimate various ATSMs. They reported identification problems in the canonical forms from Dai and Singleton (2000).

Closer attention has been paid to the problems that arise in the estimation of ATSMs in recent studies. Hamilton and Wu (2012) study the canonical representations of Gaussian ATSMs finding that they allow for multiplicity of global optima and flattening of the loglikelihood function in the presence of unit-roots. They demonstrated how the parameters reported in renowned studies within the ATSMs literature correspond to local maxima and propose an alternative representation with stronger restrictions.

Joslin et al. (2011) developed yet another specification for Gaussian dynamic term structure models. Assuming an observable state vector consisting of yield portfolios, they reported improvements in the estimation procedure. They also discussed implications of imposing no-arbitrage on yield forecasts.

A different specification that has been recently studied for ATSMs is the one proposed by Christensen et al. (2011). They matched the implied ATSM factor loadings for yields with Nelson-Siegel factor loadings. This study conciliated the ATSMs framework with dynamic Nelson-Siegel models, which have also been widely used for modeling the yield curve since their conception by Diebold and Li (2006). The imposition of Nelson-Siegel loadings on ATSMs reportedly improves the tractability of the estimation procedure.

In the Colombian setting, most approaches to dynamically modeling the TS have used the Nelson-Siegel family of models. For instance, Maldonado-Castaño et al. (2014) used the Kalman filter to estimate the underlying factors of a dynamic Nelson-Siegel model using Colombian data. Although factor estimates differed considerably from those reported by *Infoval* (Colombian price provider), the zero-coupon yield curves obtained a close fit.

Melo-Velandia and Castro-Lancheros (2010) adopted the dynamic Nelson-Siegel model from Diebold et al. (2006) to relate the Colombian yield curve dynamics with macroeconomic factors. Their model includes macro-factors as variables in the state vector which affect the latent components but do not appear directly in the yield equation. The considered macro-factors were: the interbank rate, the emerging markets bond index for Colombia, the consumer price index and the GDP gap.

IV. Justification

Modeling and understanding the TS is beneficial from various perspectives. Market participants use it to price financial instruments, take investment decisions, and manage risk. Consumers can use it to take saving or consumption decisions. The TS also contains information about the current and future states of financial markets and the economy, which can be used by policymakers.

These possibilities expand when researchers advance from curve-fitting or purely autoregressive models to models with a theoretical foundation, as are ATSMs. These models allow for the study of important unobservable variables which they incorporate, such as the short rate and market prices of risk. In this sense, we believe the Colombian financial market could benefit as a whole from the diffusion of studies that use ATSMs to model the Colombian yield curve. This process recently started with Vásquez-Galindo (2015) and Velásquez-Giraldo and Restrepo-Tobón (2016).

ATSMs can be further expanded by including macroeconomic factors in their formulation. The most popular approach is to add the macro-factors as entries in the state vector $X(t)$. This allows researchers to identify relationships between the latent factors that drive the yield curve (usually associated with level, slope and curvature) and macroeconomic variables. The relationship between specific macro-factors and risk premiums for different maturities can also be drawn using these models. A study of this nature using the model from Diebold et al. (2006) was carried out in Colombia by Melo-Velandia and Castro-Lancheros (2010).

However, there are no studies that use macro-factor ATSMs to model the Colombian yield curve (to the best of our knowledge). Furthermore, the existing literature has focused on the low-frequency dynamics of the yield curve (using monthly, quarterly or yearly data). We are interested in the daily dynamics of the Colombian yield curve, which is problematic as yields become highly persistent and daily macroeconomic time series are scarce. This creates an interesting point in evaluating whether the traditional factor dynamics (level, slope, curvature) hold at a daily frequency and testing different macro-factors from the ones that are commonly used.

V. Project Scope

Out of the great variety of dynamic models that have been proposed for describing the TS, this project will focus on a three-factor Gaussian ATSM. The baseline model will be modified to incorporate macroeconomic variables. We will only consider modifications that maintain the possibility of estimating the models using the

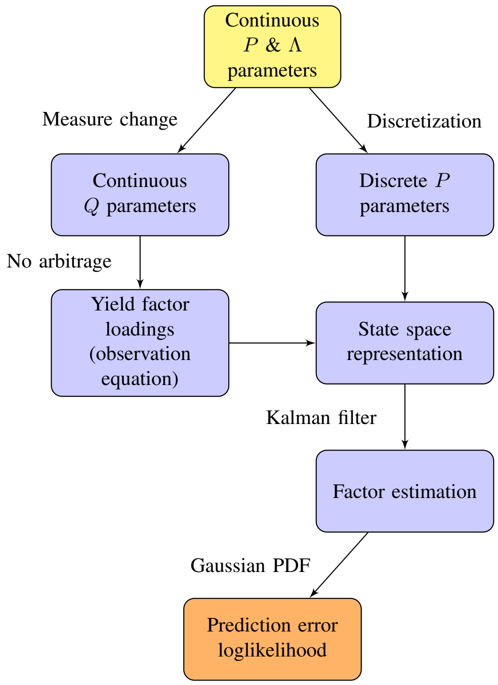

Figure 1. Obtention of the loglikelihood function to be minimized for a given set of parameters.

Kalman filter or the methodology proposed by Hamilton and Wu (2012).

The analysis of results will be limited to evaluating which of the model modifications are the most successful. Success will be assessed on the basis of estimation convergence, in-sample fit, and forecasting capabilities.

VI. Methodology

We will work with a dataset of unsmooth zero-coupon Colombian yields obtained from Bloomberg in the time period from April 2005 to May 2015. The macroeconomic variables to be included haven't been defined yet, but their time series will also be obtained from Bloomberg. Some preliminary posible variables are: the Colombian inter-bank rate, the emerging markets bond index (EMBI) for Colombia, and the USD/COP exchange rate.

State dynamics for ATSMs can be specified in continuous time (as a stochastic differential equations) or in discrete time (as vector auto-regressive processes). We have found problems in obtaining yield factor loadings in discrete models when the yield maturities are much longer than the sampling period. Therefore, we will work with a continuous specification for obtaining the loadings and discretize the model to obtain a state-space representation. This approach is based in the work of Christensen et al. (2011), who apply this methodology to an arbitrage-free Nelson-Siegel model.

Activity / Objective to be met	Time range
Implement and estimate the baseline model	Feb 12 - Mar 10
Design and test macro-factor models	Mar 10 - Apr 15
Produce forecasts & analyze results.	Apr 15 - May 15
Elaborate the final report	May 15 - May 20

Table I
Activity schedule.

With the state-space representation, we will find the unobservable factors using the Kalman filter and estimate the model parameters maximizing the prediction error log-likelihood. The procedure is summarized in Figure 1. Once the estimation methodology is defined and tested, we will start altering the baseline model to include macroeconomic factors. Two important aspects of the models must be kept in mind when the modifications take place:

- The models must remain Gaussian for the estimation procedure to be applicable (because of the Kalman filter's assumptions).
- The resulting models must be identifiable. This means that different parameter values must generate different implied distributions for the observations.

After estimating the models, they will be used to produce daily forecast of yields with different maturities. They will be ranked according to their root mean squared errors (RMSEs). The factor loadings of the best models will be examined over different maturities to analyze their impact on different parts of the yield curve. It's of special interest to check whether the traditional level, slope and curvature interpretations persist and if any of the latent factors can be replaced by an observable macroeconomic variable.

VII. Activity Schedule

An estimated schedule of the different phases of the project is presented in Table I. As the project is enclosed within the 'Research practise 3' course from the Mathematical Engineering program, various documents and presentations must be elaborated. The current[2] deadlines for these reports and presentations are shown in Table II.

VIII. Budget

The project won't require any financing. However, it should be noted that we will be using the following resources from EAFIT University:

- Matlab and Bloomberg licenses.

[2]Subject to change. Taken from: http://www1.eafit.edu.co/asr/courses/research-practises-me/2016-1/index.html.

Report / Presentation	Deadline
Project proposal report	Feb 12
Project proposal presentation	Feb 29
Progress presentation	Apr 8
Final report	May 20
Final presentation	June 7

Table II
DATES FOR REPORTS AND PRESENTATIONS.

- Subscriptions to academic data-bases.
- Tutor's working hours.

IX. INTELLECTUAL PROPERTY

In accordance with EAFIT University's intellectual property ruling (EAFIT University (2009)), the patrimonial rights over all the academic products resulting from this project will belong to:

- Mateo Velásquez-Giraldo.
- Diego Alexander Restrepo-Tobón.
- Universidad EAFIT.

The ruling also states that utilities obtained through commercialization of any product of the project must be divided in the following proportions:

- Mateo Velásquez Giraldo: 25%.
- Diego Alexander Restrepo Tobón: 20%.
- Universidad EAFIT: 55%.

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