COMPUTATIONAL IMPLEMENTATION OF THE CALCULATION OF SOME INTEGRALS RELATED TO THE WAVELET-GALERKIN METHOD

Research practise 2 project presentation

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Introduction

Precedents: Underlying theory

Calculus of variations

Is the branch of mathematics concerned with the problem of finding a function for which the value of a certain integral (functional) is either the largest or the smallest possible. Many problems of this kind are easy to state, but their solutions often entail difficult procedures of differential calculus.[1]

Variational principle

"A scientific principle used within the calculus of variations, which develops general methods for finding functions which extremize the value of quantities that depend upon those functions".[2]

[1] Extracted from web-posted Encyclopœdia Britannica article "Calculus of Variations".

[2] *Varional principle*. In *Wikipedia*. Retrieved February 16, 2016, from `https://en.wikipedia.org/wiki/Variational_principle`.

Introduction

Precedents: Functions and Functionals

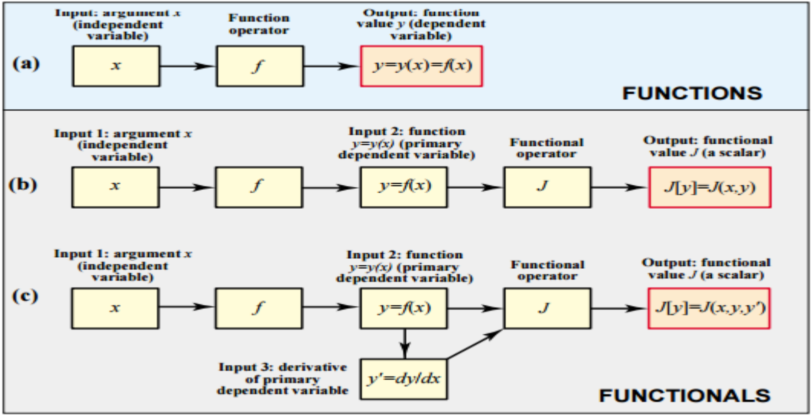

Figure 1: Block diagrams that illustrate key differences between functions and functionals in one dimension. (a) An ordinary function $y = y(x) = f(x)$ of the independent variable x; (b) a functional $J[y] = J(x, y)$ of the function $y(x)$; (c) a functional $J[y] = J(x, y, y')$ of the function $y(x)$ and its derivative $y' = dy/dx$.[1]

Examples

- Curves of Shortest Length — Planar Geodesics.
- Minimal Surface of Revolution.
- The Brachistochrone Problem.
- Isoparametrics problems.
- Mean weighted residuals and method.
- Quantum physics.
- Galerkin method

Definition

Problem: Description

Variational main problem

Let L be a differential operator defined on a dense set $\mathcal{D}_L$ over the Hilberth space $L^2([0,1])$ and f a given function. The problem is to find an approximate solution of the equation

$$\begin{aligned} &Lu = f \quad \text{on} \quad \mathcal{D}_L. \\ &\text{with Dirichlet boundary conditions} \quad u(0) = a, u(1) = b. \end{aligned} \tag{1}$$

Definition

Problem: Galerkin method

Conceptual description

Method for solving differential equation defined using functionals by the determination of several coefficients in a power series solution of the differential equation so that the functional is orthogonal to every solution approximation.[3]

[3] Weisstein, Eric W. "Galerkin Method". From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/GalerkinMethod.html.

Formulations

Mathematical formulation

Consider ϕ_i as a base of $L^2([0,1])$ and every ϕ_i satisfying C^2 on $[0,1]$ such that $\phi_i(0) = a, \phi_i(1) = b$, u_0 as an approximate solution of the equation with Λ, S as a finite set of indices i and the subspace $span\{\phi_i : i \in \Lambda\}$ respectively so that [5]:

$$\langle\, Lu_0 - f, \phi_i \rangle = 0, \quad \forall i \in \Lambda \tag{2}$$

$$u_0 = \sum_{k \in \Lambda} a_k \phi_k \in S. \tag{3}$$

Letting $\tilde{u}$ of the form (3) as the approximate solution of (2) it is intended that the residue $R = L\tilde{u} - f$ to be orthogonal to the chosen base on $\mathcal{D}_L$.

Formulations

Problem: Wavelet-Galerkin method

The Wavelet-Galerkin method considers $\phi(x) = \Psi_{j,k}(x) = 2^{j/2}\Psi(2^j x - k)$ as a wavelet basis for $L^2([0,1])$ satisfying the boundary conditions $\Psi_{j,k}(0) = \Psi_{j,k}(1) = 0$ and $\forall j, k \in \Lambda$ then $\Psi_{j,k}$ is C^2. Using the Daubechies [3] wavelets then both $a_{j,k}$ and $\Psi_{j,k}$ can be computed setting the scaling and mother wavelet functions respectively as

$$\begin{aligned} \varphi(x) &= \sum_{k=0}^{L-1} a_k \varphi(2x - k) \\ \Psi(x) &= \sum_{k=2-L}^{L} (-1)^k p_{1-k} \varphi(2x - k). \end{aligned} \tag{4}$$

Formulations

Daubechies Wavelets

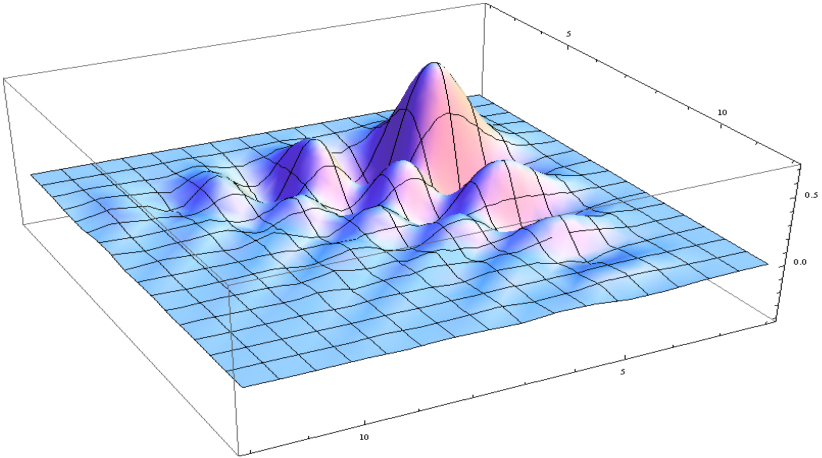

Figure 2: Daubechies 20 2-d wavelet (Wavelet Fn X Scaling Fn)

Formulations

Problem: Wavelet-Galerkin method

It is neccesary to computate several expressions in order to find the solution of differential equation by using this method, specifically the *Connection coefficients* [4] defined as follows:

Connection coefficients

$$\Omega_{j,k}^{m,n}(x) = \int_0^x \varphi(y)\varphi^{(m)}(y-j)\varphi^{(n)}(y-k)\mathrm{d}y.$$

Besides, it is also required to evaluate the following integrals [2].

Formulations

Problem: Wavelet-Galerkin method

Adjacent measures

$$\varphi^{(n)}(x) = \frac{\mathrm{d}^n \varphi(x)}{\mathrm{d}x^n},$$

$$\theta_n(x) = \underbrace{\int_0^x \int_0^{y_n} \cdots \int_0^{y_2}}_{n-tuple} \varphi(y_1)\mathrm{d}y_1 \cdots \mathrm{d}y_{n-1}\mathrm{d}y_n,$$

$$M_k^m(x) = \int_0^x y^m \varphi(y-k)\mathrm{d}y,$$

$$\Gamma_k^n(x) = \int_0^x \varphi^{(n)}(y-k)\varphi(y)\mathrm{d}y,$$

$$\Lambda_k^{m,n}(x) = \int_0^x y^m \varphi^{(n)}(y-k)\varphi(y)\mathrm{d}y,$$

$$\Upsilon_k^{m,n}(x) = \int_0^x y^m \theta_n(y-k)\varphi(y)\mathrm{d}y$$

Project

Objectives and Schedule

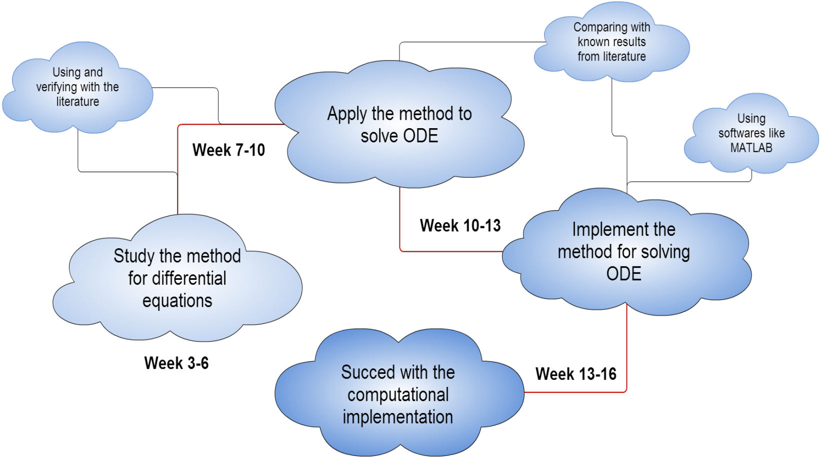

Figure 3: Project objectives and schedule

Acknowledgment

THANK YOU FOR YOUR ATTENTION!

QUESTIONS?

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[5] MISHRA, V. AND SABINA, *Wavelet Galerkin Solutions of Ordinary Differential Equations*. International Journal of Mathematics, vol. 5, no. 9, pp. 407–424 (2011).