Computational implementation of the calculation of some integrals related to the Wavelet-Galerkin method
Research practise 2 progress presentation

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Formulations

Mathematical formulation

Consider ϕ_i as a base of $L^2([0,1])$ and every ϕ_i satisfying C^2 on $[0,1]$ such that $\phi_i(0) = a, \phi_i(1) = b$, u_0 as an approximate solution of the equation with Λ, S as a finite set of indices i and the subspace $span\{\phi_i : i \in \Lambda\}$ respectively so that [3]:

$$\langle\, Lu_0 - f, \phi_i \rangle = 0, \quad \forall i \in \Lambda \tag{1}$$

$$u_0 = \sum_{k \in \Lambda} a_k \phi_k \in S. \tag{2}$$

Letting $\tilde{u}$ of the form (2) as the approximate solution of (1) it is intended that the residue $R = L\tilde{u} - f$ to be orthogonal to the chosen base on $\mathcal{D}_L$.

Formulations

Problem: Wavelet-Galerkin method

The Wavelet-Galerkin method considers $\phi(x) = \Psi_{j,k}(x) = 2^{j/2}\Psi(2^j x - k)$ as a wavelet basis for $L^2([0,1])$ satisfying the boundary conditions $\Psi_{j,k}(0) = \Psi_{j,k}(1) = 0$ and $\forall j,k \in \Lambda$ then $\Psi_{j,k}$ is C^2. Using the Daubechies [1] wavelets then both $\varphi_{j,k}$ and $\Psi_{j,k}$ can be computed setting the scaling and mother wavelet functions respectively as

$$\begin{aligned} \varphi(x) &= \sum_{k=0}^{L-1} a_k \varphi(2x - k) \\ \Psi(x) &= \sum_{k=2-L}^{L} (-1)^k a_{1-k} \varphi(2x - k), \end{aligned} \tag{3}$$

with a_k characterized for the N-Daubechies wavelet grade.

Calculations

Daubechies Wavelets

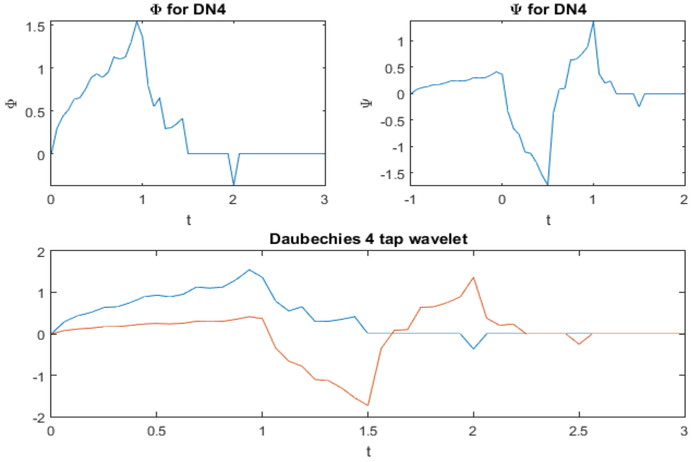

Figure 1: Scaling and Wavelet functions for DN4

Formulations

Daubechies Wavelets

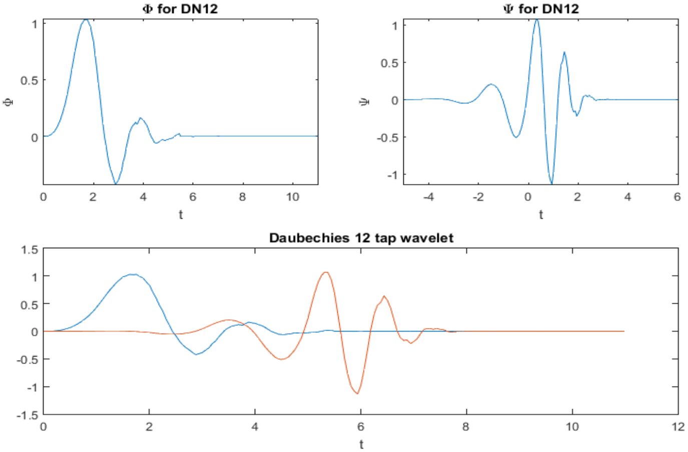

Figure 2: Scaling and Wavelet functions for DN12

Calculations

Daubechies Wavelets

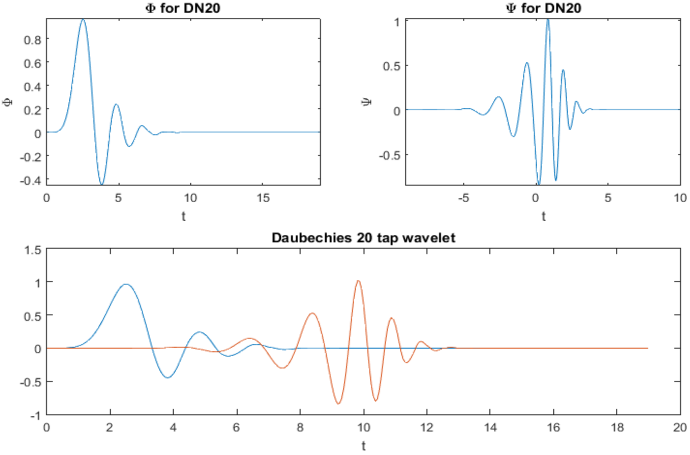

Figure 3: Scaling and Wavelet functions for DN20

It is neccesary to computate several expressions in order to find the solution of differential equation by using this method, specifically the *Connection coefficients* [2] defined as follows:

Connection coefficients

$$\Omega_{j,k}^{m,n}(x) = \int_{-\infty}^{\infty} \varphi^{(m)}(y-j)\varphi^{(n)}(y-k)\mathrm{d}y. \tag{4}$$

Taking the respective derivatives and simplificating the following system of linear equations is found, where $\Omega^{m,n}$ is the unknown vector to be calculated.

$$\begin{pmatrix} T - \frac{1}{2^{d-1}} I \\ M^d \end{pmatrix} \Omega^{m,n} = \begin{pmatrix} 0 \\ d! \end{pmatrix} \tag{5}$$

where $d = m + n$, $T = \sum_i a_i a_{q-2l+i}$ and M_i^k are the moments of φ_i defined as

$$M_i^k = \int_{-\infty}^{\infty} x^k \varphi_i(x) dx,$$

satisfying $M_0^0 = 1$.

Calculations

Connection Coefficients Calculations

$\Omega[-4]$	$5.357142857141725e-03$
$\Omega[-3]$	$1.142857142857160e-01$
$\Omega[-2]$	$-8.761904761904885e-01$
$\Omega[-1]$	$3.390476190476218e+00$
$\Omega[0]$	$-5.267857142857142e+00$
$\Omega[1]$	$3.390476190476168e+00$
$\Omega[2]$	$-8.761904761904653e-01$
$\Omega[3]$	$1.142857142857138e-01$
$\Omega[4]$	$5.357142857143558e-03$

Table 1: Connection Coefficients for
$N = 6$, $j = 0$, $m = 2$, $n = 0$

Calculations

Connection Coefficients Calculations

$\Omega[-4]$	$8.777142857143009e+01$
$\Omega[-3]$	$1.872457142857140e+03$
$\Omega[-2]$	$-1.435550476190474e+04$
$\Omega[-1]$	$5.554956190476182e+04$
$\Omega[0]$	$-8.630857142857110e+04$
$\Omega[1]$	$5.554956190476169e+04$
$\Omega[2]$	$-1.435550476190469e+04$
$\Omega[3]$	$1.872457142857137e+03$
$\Omega[4]$	$8.777142857143159e+01$

Table 2: Connection Coefficients for $N = 6$, $j = 7$, $m = 2$, $n = 0$

Calculations

Connection Coefficients Calculations

$\Omega[-6]$	$2.547463883891842e-04$
$\Omega[-5]$	$-2.608603017123225e-02$
$\Omega[-4]$	$-1.691636444481563e-01$
$\Omega[-3]$	$2.415566393856456e+00$
$\Omega[-2]$	$-1.116590566972836e+01$
$\Omega[-1]$	$4.227312332967440e+01$
$\Omega[0]$	$-6.665557825114264e+01$
$\Omega[1]$	$4.227312332967382e+01$
$\Omega[2]$	$-1.116590566972807e+01$
$\Omega[3]$	$2.415566393856424e+00$
$\Omega[4]$	$-1.691636444481482e-01$
$\Omega[5]$	$-2.608603017123400e-02$
$\Omega[6]$	$2.547463883939815e-04$

Table 3: Connection Coefficients for
$N = 8$, $j = 2$, $m = 2$, $n = 0$

Formulations

2-term Connection Coefficients

Let us consider the general integral-differential equation depending on u with $x \in [a, b]$:

$$f(x, \tfrac{du}{dx}, \tfrac{d^2u}{dx^2}, \ldots, \int^x u dx_1, \int^x \int^{x_1} u dx_2 dx_1, \ldots) = 0. \tag{6}$$

Following the common notation for the approximation of u according to (2), we have $\tilde{u}$ is as follows:

$$\tilde{u}(x) = \sum_{k=1-L}^{2^j} c_k \varphi_{j,k}(x) = \sum_{k=1-L}^{2^j} c_k 2^{j/2} \varphi(2^j x - k). \tag{7}$$

Formulations

2-term Connection Coefficients

Using this approximation, the coefficients c_k are determined by applying the inner product and solving (8) for $k = 1 - L, \ldots, 2^j$.

$$\int_a^b \varphi_{j,k}(x) f(x, \tfrac{d\tilde{u}}{dx}, \tfrac{d^2\tilde{u}}{dx^2}, \ldots, \int^x \tilde{u} dx_1, \int^x \int^{x_1} \tilde{u} dx_2 dx_1, \ldots) = 0 \quad (8)$$

Application

Example

Consider the problem

$$\frac{d^2u}{dx^2} + \beta u = 0, \quad 0 < x < 1, \\ u(0) = 1 \quad \text{y} \quad u(1) = 0. \tag{9}$$

Whose exact solution is $u(x) = \cos(x) - \cot(1)\sin(x)$.

When solving this second-order linear differential equation through the Wavelet-Galerkin method concerning u as unknown function then the two terms-connection coefficients result from (4) for $n = 1 - L, \ldots, 2^j$ into

$$\Omega_{j,k}^{m,n}(x) = \Omega_k^n = \Omega[n-k] = \int_{-\infty}^{\infty} \varphi_k^{''}(x)\varphi_n(x)dx.$$

Application

Example $L = 6$ and $j = 0$

According to (8) we must find c_k such that

$$\sum_{k=-5}^{1} c_k \Omega[n-k] + \beta \sum_{k=-5}^{1} c_k \delta_{n,k} = 0, \quad \text{where}$$

$$\Omega[n-k] = \int \varphi''(x-k)\varphi(x-n) \quad \text{and}$$

$$\delta_{n,k} = \int \varphi(x-k)\varphi(x-n).$$

Using the coefficients for this case Ω_k^n we build the following linear system $TC = B$, where C is the unknown vector $C^T = \begin{pmatrix} c_{-5} & c_{-4} & c_{-3} & c_{-2} & c_{-1} & c_0 & c_1 \end{pmatrix}$ and

Application

Example $L = 6$ and $j = 0$

$$T = \begin{pmatrix} 0 & \varphi(4) & \varphi(3) & \varphi(2) & \varphi(1) & 0 & 0 \\ \Omega[1] & \Omega[0]+\beta & \Omega[-1] & \Omega[-2] & \Omega[-3] & \Omega[-4] & \Omega[-5] \\ \Omega[2] & \Omega[1] & \Omega[0]+\beta & \Omega[-1] & \Omega[-2] & \Omega[-3] & \Omega[-4] \\ \Omega[3] & \Omega[2] & \Omega[1] & \Omega[0]+\beta & \Omega[-1] & \Omega[-2] & \Omega[-3] \\ \Omega[4] & \Omega[3] & \Omega[2] & \Omega[1] & \Omega[0]+\beta & \Omega[-1] & \Omega[-2] \\ \Omega[5] & \Omega[4] & \Omega[3] & \Omega[2] & \Omega[1] & \Omega[0]+\beta & \Omega[-1] \\ 0 & 0 & \varphi(4) & \varphi(3) & \varphi(2) & \varphi(1) & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Application

Example $L = 6$ and $j = 0$

Solving the last system we find

$$C = \begin{pmatrix} -0.9972 \\ -0.8776 \\ 0.1279 \\ 1.0543 \\ 1.0870 \\ 0.2479 \\ -0.5059 \end{pmatrix}$$

and therefore

$$\tilde{u}(x) = \sum_{k=-5}^{1} c_k \varphi(x-k) = -0.9972\varphi(x+5) - 0.8776\varphi(x+4) + \ldots$$

$$0.1279\varphi(x+3) + 1.0543\varphi(x+2) + 1.0870\varphi(x+1) + \ldots$$

$$0.2476\varphi(x) - 0.5059\varphi(x-1).$$

Calculations

Daubechies Wavelets

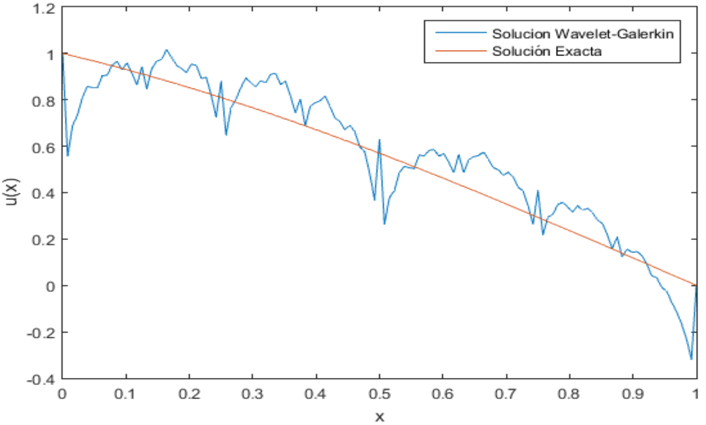

Figure 4: Exact and approximate solution of $u'' + u = 0$, with $u(0) = 1$ and $u(1) = 0$.

Calculations

Daubechies Wavelets

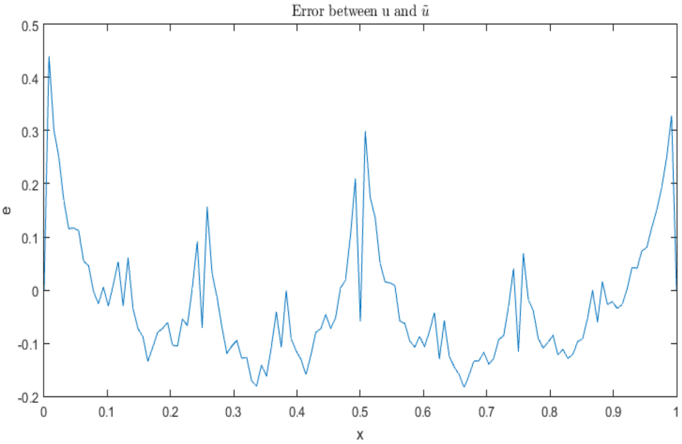

Figure 5: Error between u and $\tilde{u}$

Project

Where are we now?

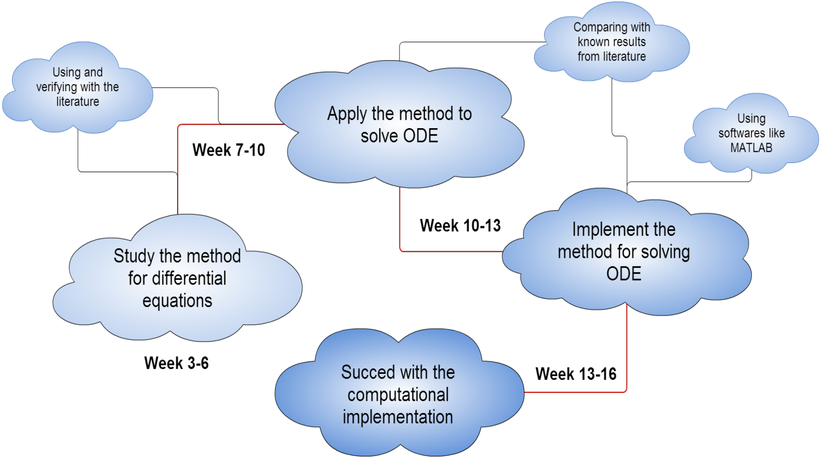

Figure 6: Project objectives and schedule

Acknowledgment

THANK YOU FOR YOUR ATTENTION!

QUESTIONS?

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