Application of the Wavelet-Galerkin Method on the homogeneous second order linear ordinary differential equation with constant coefficients

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***Abstract*—In this work we present the methodology and results coming from the application of the *Wavelet-Galerkin* method on a second order linear ordinary differential equation with constant coefficients. The classical scaling and mother wavelet functions are used to find a proper approximation for the studied case. Through this article several algorithms are displayed in order to make a posterior implementation of the method. Finally, the approximate curve and errors for a particular case are analyzed using the proposed methodology to show the method potential and behavior.**

***Keywords*—Wavelet-Galerkin method, connection coefficients, numerical approximation, ordinary differential equations**

I. Introduction

The development of methods able to result into approximations of differential equations, ordinary and partial, is certainly a subject of general interest in the industry and academic society since usually they describe processes underlying real phenomena whose mathematical description and simulation are generally use to increase its profitableness or to understand its behavior; nevertheless, these approximations usually seek to satisfy certain conditions related to the good behavior of the solution as the time increases or to the initial and boundary conditions imposed. Therefore, these methods must be improved or new others introduced so that an equilibrium between performance accuracy and computational resources can be reached.

The wavelets are a mathematical object whose properties, such as orthogonality, compact support, good representation of polynomials and ability to represent functions at different levels of resolution [2] are desirable and useful when dealing with basis for a Galerkin approach to solve differential equations. The study of techniques based on wavelets has let the introduction of a new family of numerical methods whose computational requirements are considerably less than those by finite differences and finite elements; nonetheless, the implementation of such methods is not arbitrary and depends upon whether it is computable or not and the quality of the discretized scheme. By the exploitation of the properties of the wavelet basis functions one wants to find simpler ways to solve the integral operations associated to the method [10] coming from the need to determinate the values of inner products required to evaluate the method results.

The wavelet yield to the construction of mathematical models similar to those coming from the variational calculus, which deals with maximization or minimization of functionals, and have generated a great interest due to its properties. On one side, the theorical research on this subject has increased widely since the introduction of the *Haar* system [3], due to the Hungarian mathematician *Alfred Haar* around 1910, which allowed to represent a target function over an interval in terms of an orthogonal function basis, but it had the disadvantage of not being continuous and therefore not differential, both essential conditions to be satisfied in further and more complex analysis. Later in 1988, the Belgian physicist *Ingrid Daubechies* proposed the construction of orthogonal wavelets with compact support and proves some of their most relevant properties, including the possibility to compute their connection coefficients. In terms of the *Galerkin* method, wavelet basis functions are considerably practical as a result of their characteristics and therefore the test function is considered as a linear combination of a wavelet basis.

On the other hand, as a mathematical tool, wavelets can be used to extract information from many different kinds of data, to compress it and are also used to represent a wide variety of curves and surfaces in CADD (computer-aided design and drafting) softwares, hence are of great interest in Applied mathematics and Engineering [9]. Thus, in terms of practical applications and theorical use potential the improvement and exploration of this method, especially as capable to produce enhancement on the numerical methods to approximate the solution of differential equations, is of high significance.

When applying this last method to solve ordinary or partial differential equations some improper integrals, known as connection coefficients, result as terms in its equations, but when it is related to a bounded interval then produces proper connection coefficients [10]. Either way, it is necessary to compute these coefficients, whose analytical solution has not been found and whose numerical one requires the use of complex algorithms in order to approximate them. Thus, several techniques have

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been proposed, which take advantage of the nature of the basis functions to build series approximations for later calculations via algorithms for several functionals. In a first stage, *Latto et al.* [4], give a first approach to the calculation of these coefficients by imposing periodic boundary conditions and deriving a formula to compute the moments by induction but he provides connection coefficients only for two cases. Later, *Mishra* [7], *Lin* [5], *Popovici* [8] and others worked on the same subject and were able to extend the computations for all the cases under different techniques.

II. Problem statement

Let L be a differential operator defined on the real numbers set over the $L^2([0,1])$ space, which denotes the *Hilbert* space of square integrable functions on $[0,1]$ and f a given function. The problem is to find an approximate solution of the equation

$$\begin{aligned} &Lu = f \quad \text{on} \quad \mathbb{R}, \\ &\text{with } \textit{Dirichlet} \text{ boundary conditions} \\ &u(0) = a, \quad u(1) = b, \quad \{a,b\} \in \mathbb{R} \end{aligned} \tag{1}$$

The Galerkin technique consists of considering ϕ_i as a base of $L^2([0,1])$ and every ϕ_i satisfying C^2 on $[0,1]$ such that $\phi_i(0) = a$, $\phi_i(1) = b$, u_0 with Λ as a finite set of indices i and S the subspace $span\{\phi_i : i \in \Lambda\}$ so that

$$\langle\, Lu_0 - f, \phi_i \rangle = 0, \quad \forall i \in \Lambda \tag{2}$$

then it follows that $Lu_0 - f = 0$ in $L^2([0,1])$ and therefore u_0 is a solution for the problem (1). Letting $\tilde{u}$ be the approximate solution of the given problem of the form (3) such that (2) is satisfied, that is to make the residue $R = L\tilde{u} - f$ to be orthogonal to the chosen base on $\mathbb{R}$.

$$\tilde{u} = \sum_{k \in \Lambda} a_k \phi_k \tag{3}$$

The Galerkin-Wavelet method takes $\Psi_{j,k}(x) = \phi(x) = 2^{j/2}\Psi(2^j x - k)$ as a wavelet basis for $L^2([0,1])$ satisfying the boundary conditions $\Psi_{j,k}(0) = \Psi_{j,k}(1) = 0, \forall j,k \in \Lambda$. Such $\Psi_{j,k} \in \mathbb{R}$ is defined as a family of functions through scaling by a certain factor of a function $\Psi \in L^2(\mathbb{R})$. Now, replacing the last expression into (3) then

$$\tilde{u} = \sum_{j,k \in \Lambda} a_{j,k} \Psi_{j,k}. \tag{4}$$

Notice that (2) can also be written as

$$\begin{aligned} &\langle\, L\tilde{u}, \phi_i \rangle = \langle\, f, \phi_i \rangle, \quad \forall i \in \Lambda, \quad \text{substituing } \tilde{u} \\ &\sum_{j,k \in \Lambda} \langle\, L\Psi_{j,k}, \Psi_{l,m} \rangle a_{j,k} = \langle\, f, \Psi_{l,m} \rangle, \quad \forall l,m \in \Lambda \end{aligned}$$

As commented before, the *Daubechies* wavelets play a fundamental role in the construction of $a_{j,k}$ and $\Psi_{j,k}$ since they form a compactly supported orthonormal base which includes members from highly localized to highly smooth frequency. Such wavelets are determined by a set of N coefficients (genus of *Daubechies* wavelet) $\{p_k : k = 0, 1, \ldots, N-1\}$, p_k denoted as wavelet filter coefficients, through the relation

$$\varphi(x) = \sum_{k=0}^{N-1} \sqrt{2} a_k \varphi(2x - k), \quad supp(\varphi) = [0, N-1] \tag{5}$$

and the equation

$$\Psi(x) = \sum_{k=2-N}^{1} (-1)^k \sqrt{2} a_{1-k} \varphi(2x - k), supp(\Psi) = [1 - \tfrac{N}{2}, \tfrac{N}{2}] \tag{6}$$

where $\varphi(x)$ and $\Psi(x)$ are called scalling function and mother wavelet respectively. The factor $\sqrt{2}$ is conventionally used to assure that both (5) and (6) are normalized to have sum $\sqrt{2}$ by taking $\sum_{i=0}^{N-1} a_i = \sqrt{2}$.

In order to compute this last pair of functions for $k \in [0, N-1]$ and $k \in [2-N, 1]$ respectively it is necessary, in both cases, to count with the $\boldsymbol{a} = \{a_0, a_1, \ldots, a_{N-1}\}$ *Daubechies* coefficients and N initial values for $\varphi(x)$, specifically in $x = 0, 1, 2, \ldots, N-1$. These initial values are the components of the corresponding eigenvector for the eigenvalue 1 coming from the system formed after evaluating the scaling function for those values.

For instance, for $N = 6$, then (5) and (6) assume the folowing recursive form:

$$\begin{aligned} &\varphi(x) = \sum_{k=0}^{5} \sqrt{2} a_k \varphi(2x - k), && supp(\varphi) = [0, 5] \\ &\Psi(x) = \sum_{k=-4}^{1} (-1)^k \sqrt{2} a_{1-k} \varphi(2x - k), && supp(\Psi) = [-2, 3] \end{aligned}$$

Evaluating for different values of x in the $\varphi-$support and having in mind that in this case $\varphi(x) = 0$ for $x \notin supp(\varphi)$ then:

$$\begin{aligned} x = 0, \quad & \varphi(0) = \sqrt{2}(a_0\varphi(0)) \\ x = 1, \quad & \varphi(1) = \sqrt{2}(a_0\varphi(2) + a_1\varphi(1) + a_2\varphi(0)) \\ x = 2, \quad & \varphi(2) = \sqrt{2}(a_0\varphi(4) + a_1\varphi(3) + a_2\varphi(2) \\ & \qquad + a_3\varphi(1) + a_4\varphi(0)) \\ x = 3, \quad & \varphi(3) = \sqrt{2}(a_2\varphi(5) + a_3\varphi(4) + a_4\varphi(3) \\ & \qquad + a_5\varphi(2)) \\ x = 4, \quad & \varphi(4) = \sqrt{2}(a_3\varphi(5) + a_4\varphi(4) + a_5\varphi(3)) \\ x = 5, \quad & \varphi(5) = \sqrt{2}(a_5\varphi(5)) \end{aligned} \tag{7}$$

This way the system of linear equations with the vector $\boldsymbol{\varphi} = [\varphi(0), \varphi(1), \ldots, \varphi(5)]$ as unknown variables is formed:

$$\boldsymbol{\varphi} = \sqrt{2}\begin{bmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 & 0 & 0 \\ a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & a_5 & a_4 & a_3 & a_2 & a_1 \\ 0 & 0 & 0 & a_5 & a_4 & a_3 \\ 0 & 0 & 0 & 0 & 0 & a_5 \end{bmatrix}\boldsymbol{\varphi}$$
$$\Rightarrow \lambda\boldsymbol{\varphi} = A\boldsymbol{\varphi}$$

This system corresponds to an eigenvalue problem with $\lambda = 1$ and A as the coefficients matrix, thus by solving it further values of the scaling and mother wavelet function can be computated.

This way, depending on the scale and wavelet order, the scaling and mother wavelet function acquires different shapes; nevertheless, the *Daubechies* wavelets are not possible to write down in a closed form, hence it is required to implement a recursive method as the following to computate them.

Algorithm 1 φ and Ψ computation

Read N, scale

$\boldsymbol{a} \leftarrow$ daubechiesCoffs
$\boldsymbol{\varphi}_0 \leftarrow$ initialValues
$x \leftarrow 0$
cont $\leftarrow 1$
while $x \in supp(\varphi)$ **do**
  $\boldsymbol{\varphi}_{\text{cont}} \leftarrow \texttt{varPhi}(x, N, \boldsymbol{\varphi}_0, a)$
  cont $\leftarrow$ cont+1
  $x \leftarrow x +$ scale
end while

$y \leftarrow 1 - N/2$
cont $\leftarrow 1$
while $y \in supp(\Psi)$ **do**
  $\boldsymbol{\Psi}_{\text{cont}} \leftarrow \texttt{Psi}(x, N, \boldsymbol{\varphi}_0, \boldsymbol{a})$
  cont $\leftarrow$ cont+1
  $y \leftarrow y +$ scale
end while

Algorithm 2 `Psi` delivers `value`

Read N, y, $\boldsymbol{a}$, $\boldsymbol{\varphi}_0$

temp $\leftarrow 0$
$k \leftarrow 2 - N$
while $k \in supp(\Psi)$ **do**
  temp $\leftarrow$ temp+$\sqrt{2}*(-1)^{k+1}*\boldsymbol{a}(2-k)*\texttt{varPhi}(2y-k, N, \boldsymbol{\varphi}_0, \boldsymbol{a})$
  k $\leftarrow$ k+1
end while
`value` $\leftarrow$ temp

Algorithm 3 `varPhi` delivers `value`

Read N, x, $\boldsymbol{a}$, $\boldsymbol{\varphi}_0$

temp $\leftarrow 0$
$conj \leftarrow [0, N-1] \cap \mathbb{Z}$
if $x \in conj$ **then**
  `varPhi` $\leftarrow \boldsymbol{a}(x)$
else
  $k \leftarrow 0$
  while $k \in supp(\varphi)$ **do**
    if $2x - k \in supp(\varphi)$ **then**
      temp $\leftarrow$ temp+$\sqrt{2}*\boldsymbol{a}(k+1)*\texttt{varPhi}(2x-k, N, \boldsymbol{\varphi}_0, \boldsymbol{a})$
    end if
    k $\leftarrow$ k+1
  end while
end if
`value` $\leftarrow$ temp

A series of graphs of the scaling and mother wavelet function gotten using the previous algorithms are given in the Figures for $N = 6$, $N = 12$ and $N = 20$, respectively, setting the "scale" parameters as 2^{-3}.

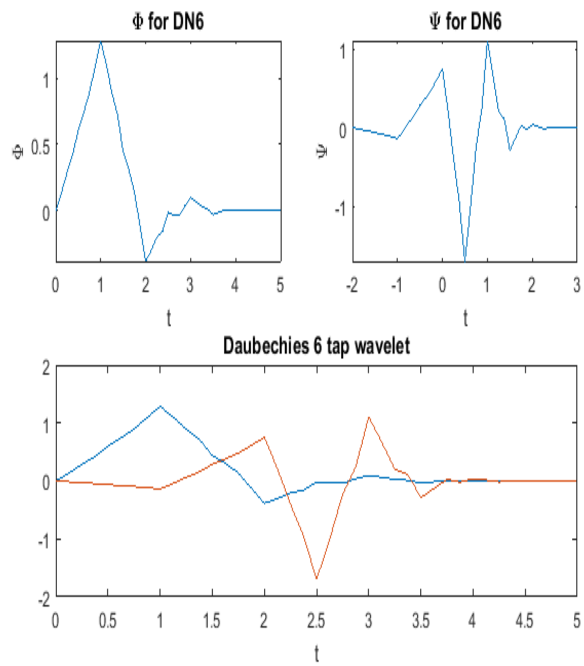

Fig. 1: Scaling and Wavelet functions for DN6

Figures 1, 2 and 3 show how, after fixing the "scale" parameter, the curves get smoother and the vanishing moments increase with the wavelet genus.

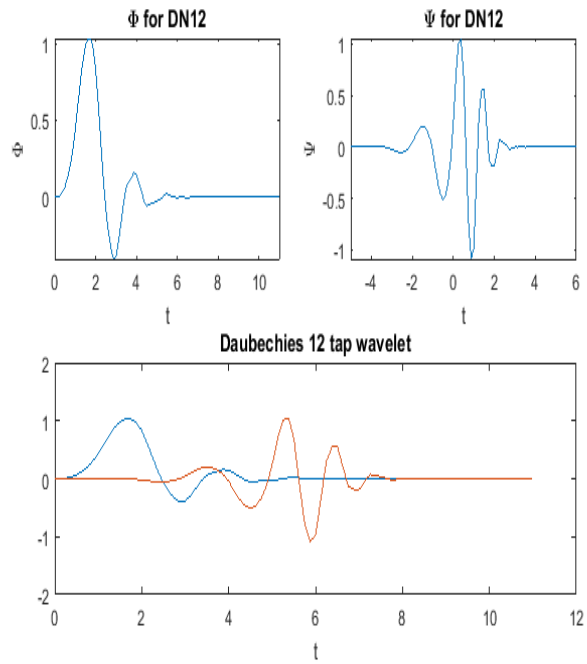

Fig. 2: Scaling and Wavelet functions for DN12

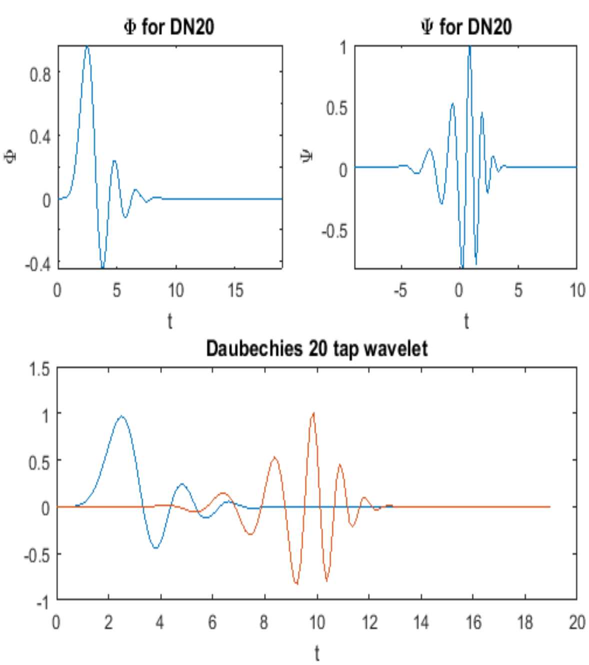

Fig. 3: Scaling and Wavelet functions for DN20

Now, if the differential operator in (1) is nonlinear with possible variable coefficients it is highly probable that the n-term connection coefficients [1], denoted as $\Omega^{d_1,\dots,d_n}_{k_1,\dots,k_n}$, come out as

$$\Omega^{d_1,\dots,d_n}_{k_1,\dots,k_n} := \int_{-\infty}^{\infty} \varphi^{d1}_{k_1} \cdots \varphi^{d_n}_{k_n} dx = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \varphi^{d_i}_{k_i} dx \tag{8}$$

where the super indices $d_1, \dots, d_n$ denote n derivatives of (5) and $\varphi_k(x) = \varphi(x-k)$ was introduced to improve the notation; however, in this article only the case where L is a second order linear differential operator with constant coefficients will be studied. Therefore, (8) turns into

$$\Omega^{d_1,d_2}_{k_1,k_2} := \int_{-\infty}^{\infty} \varphi^{d1}_{k_1}\varphi^{d_2}_{k_2} dx = \int_{-\infty}^{\infty} \varphi^{d1}(x-k_1)\varphi^{d_2}(x-k_2)dx \tag{9}$$

Notice that this integral can be taken into a simpler form by making a change of variables. Let $z = x - k_1$ in the first term in the integral, so that $x - k_2 = z - (k_1 - k_2)$ and $dz = dx$.

$$\begin{aligned} \Omega^{d_1,d_2}_{k_1,k_2} &= \int_{-\infty}^{\infty} \varphi^{d1}(z)\varphi^{d_2}(z-(k_1-k_2))dz \\ &= \int_{-\infty}^{\infty} \varphi^{d1}(z)\varphi^{d_2}(z-l)dx, \quad l = k_1 - k_2 \\ &= \Omega^{d_1,d_2}_{0,l} \end{aligned} \tag{10}$$

In order to calculate the 2-term connection coefficients for unbounded intervals *Latto et. al* [4] proposed a procedure to do so. Relying on it the bounded case is also provided on expanding the domain by adding $1 - N$ points to the left and $N - 1$ to the right. Such technique creates “Fictitious Boundaries” [6] leaving the original interval unaffected.

On one hand, let Λ^{d_1,d_2} be the column vector with the $2N-3$ connection coefficients from (10). On the other hand, taking the respective d derivatives of (5) then

$$\begin{aligned} \varphi^d &= \left(\sum_{k=0}^{N-1} \sqrt{2}a_k\varphi_k(2x)\right)^{(d)} \\ &= \left(2\sum_{k=0}^{N-1} \sqrt{2}a_k\varphi_k^{'}(2x)\right)^{(d-1)} \\ &= \cdots \\ &= 2^d \sum_{k=0}^{N-1} \sqrt{2}a_k\varphi^{(d)}_k(2x) \end{aligned} \tag{11}$$

Replacing this last equation into (10) and making the proper simplifications gives the system of linear equations:

$$\begin{pmatrix} T - \frac{1}{2^{d-1}}I \\ M^d \end{pmatrix} \Lambda^{d_1,d_2} = \begin{pmatrix} \mathbf{0} \\ d! \end{pmatrix} \tag{12}$$

where $d := d_1 + d_2$, $T_{l:q} = \sum_i a_i a_{q-2l+i}$, M^d a row vector with all the M^j_i, which are the moments of φ_i defined as

$$
\begin{aligned}
M_i^j &= \int_{-\infty}^{\infty} x^j \varphi_i(x)dx \\
&= \frac{1}{2(2^j-1)} \sum_{k=0}^{j} a \binom{j}{k} i^{j-k} \sum_{l=0}^{k-1} \binom{k}{l} M_0^l \left(\sum_{i=0}^{N-1} a_i i^{k-l} \right)
\end{aligned} \tag{13}
$$

satisfying $M_0^0 = 1$ and $d! = (-1)^d \sum_l M_l^d \Lambda_l^{0,d}$.

Notice how the connection coefficients can also be computed over the dilated and translated scaling, thus mother wavelet function by $\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$. In such case the parameter "j" is called the wavelet resolution.

In order to find the vector Λ^{d_1,d_2} the following Algorithm based on *Jordi* [1] is executed.

Algorithm 4 Λ^{d_1,d_2} computation

Read N, j, d_1, d_2
a $\leftarrow$ daubechiesCoffs
$L \leftarrow [2-N, N-2] \cap \mathbb{Z}$
T $\leftarrow$ null matrix of order $(2N-3)$
$l \leftarrow 1$
for $l = 1 : 2N-3$ **do**
  for $q = 1 : 2N-3$ **do**
    for $p = 1 : \leq N$ **do**
      index $\leftarrow L(q) - 2L(l) + p$
      if index $\in [1, N]$ **then**
        $\mathrm{T}_{l,q} \leftarrow \mathrm{T}_{l,q} + \boldsymbol{a}(p) * \boldsymbol{a}(\text{index})$
      end if
    end for
  end for
end for
Complete T with $-\frac{1}{2^{d-1}} * I$
for $i = 1 : 2N-3$ **do**
  $M_i^{d,j} \leftarrow \frac{1}{2(2^j-1)} \sum_{k=0}^{j} a \binom{j}{k} i^{j-k} \sum_{l=0}^{k-1} \binom{k}{l} M_0^l \left(\sum_{i=0}^{N-1} a_i i^{k-l} \right)$
end for
$A \leftarrow \begin{pmatrix} T - \frac{1}{2^{d-1}} I \\ M^d \end{pmatrix}$
$\Lambda^{d_1,d_2} \leftarrow A^{-1} \begin{pmatrix} \mathbf{0} \\ d! \end{pmatrix}$

The Tables I, II and III give us the 2-term connection coefficients for different instances. It is clear that as N increases, then the dimension of the vector increases as well, therefore we only present the cases $N = 6$, $N = 12$ and $N = 20$ making variations on the wavelet resolution j and on d in order to build a correspondence with the Figures (1), (2) and (3), and further applications of the method.

$\Lambda^{0,2}$	$N = 6$, $j = 0$	$N = 6$, $j = 7$
Ω_{-4}	$5.357142857141725e-03$	$8.777142857143009e+01$
Ω_{-3}	$1.142857142857160e-01$	$1.872457142857140e+03$
Ω_2	$-8.761904761904885e-01$	$-1.435550476190474e+04$
Ω_{-1}	$3.390476190476218e+00$	$5.554956190476182e+04$
Ω_0	$-5.267857142857142e+00$	$-8.630857142857110e+04$
Ω_1	$3.390476190476168e+00$	$5.554956190476169e+04$
Ω_2	$-8.761904761904653e-01$	$-1.435550476190469e+04$
Ω_3	$1.142857142857138e-01$	$1.872457142857137e+03$
Ω_4	$5.357142857143558e-03$	$8.777142857143159e+01$

TABLE I: 2-term Connection Coefficients holding $N = 6$, and $d = 2$

$\Lambda^{0,2}$	$N = 12$, $j = 0$	$N = 6$, $j = 4$
Ω_{-10}	$-1.264106326542370e-11$	$-3.23591924701283e-09$
Ω_{-9}	$2.629981125547020e-08$	$6.73275165333709e-06$
Ω_{-8}	$-3.466086045922270e-06$	$-8.87318028238585e-04$
Ω_{-7}	$-5.436337907592100e-05$	$-1.39170250435023e-02$
Ω_{-6}	$-6.569629078471730e-05$	$-1.68182504410154e-02$
Ω_{-5}	$6.478061041939120e-03$	$1.65838362673747e+00$
Ω_{-4}	$-4.936161063949490e-02$	$-1.26365723237153e+01$
Ω_{-3}	$2.049054694327110e-01$	$5.24558001747858e+01$
Ω_{-2}	$-6.307332429628940e-01$	$-1.61467710198545e+02$
Ω_{-1}	$2.311866563670020e+00$	$5.91837840299610e+02$
Ω_0	$-3.686063482147190e+00$	$-9.43632251429710e+02$
Ω_1	$2.311866563670230e+00$	$5.91837840299530e+02$
Ω_2	$-6.307332429630140e-01$	$-1.61467710198498e+02$
Ω_3	$2.049054694327380e-01$	$5.24558001747739e+01$
Ω_4	$-4.936161063950230e-02$	$-1.26365723237108e+01$
Ω_5	$6.478061041938850e-03$	$1.65838362673669e+00$
Ω_6	$-6.569629078488790e-05$	$-1.68182504409452e-02$
Ω_7	$-5.436337907648730e-05$	$-1.39170250435045e-02$
Ω_8	$-3.466086045906120e-06$	$-8.87318027686452e-04$
Ω_9	$2.629981090495320e-08$	$6.73275165380982e-06$
Ω_{10}	$-1.264104968290450e-11$	$-3.23605654884550e-09$

TABLE II: 2-term Connection Coefficients holding $N = 12$, and $d = 2$

III. Test problem

Consider the differential operator in (1) of the form

$$
\begin{aligned}
Lu &:= u'' + p_1 u' + p_2 u = 0 \\
u(0) &= a, \quad u(1) = b, \quad x \in [0,1]
\end{aligned} \tag{14}
$$

where $\{p_1, p_2, a, b\} \in \mathbb{R}$.

$\Lambda^{0,2}$	$N = 12$, $j = 0$	$\Lambda^{0,2}$	$N = 6$, $j = 4$
Ω_{-18}	$3.928343e-15$	Ω_1	$2.175217e+00$
Ω_{-17}	$-3.486099e-16$	Ω_2	$-6.066894e-01$
Ω_{-16}	$2.858395e-15$	Ω_3	$2.546974e-01$
Ω_{-15}	$-2.399663e-13$	Ω_4	$-1.054297e-01$
Ω_{-14}	$-5.015915e-11$	Ω_5	$3.758004e-02$
Ω_{-13}	$-2.219929e-09$	Ω_6	$-1.078072e-02$
Ω_{-12}	$6.114256e-09$	Ω_7	$2.357271e-03$
Ω_{-11}	$1.222971e-07$	Ω_8	$-3.693880e-04$
Ω_{-10}	$-2.579303e-06$	Ω_9	$3.852452e-05$
Ω_{-9}	$3.852452e-05$	Ω_{10}	$-2.579303e-06$
Ω_{-8}	$-3.693880e-04$	Ω_{11}	$1.222971e-07$
Ω_{-7}	$2.357271e-03$	Ω_{12}	$6.114256e-09$
Ω_{-6}	$-1.078072e-02$	Ω_{13}	$-2.219929e-09$
Ω_{-5}	$3.758004e-02$	Ω_{14}	$-5.015874e-11$
Ω_{-4}	$-1.054297e-01$	Ω_{15}	$-2.399884e-13$
Ω_{-3}	$2.546974e-01$	Ω_{16}	$2.821739e-15$
Ω_{-2}	$-6.066894e-01$	Ω_{17}	$-3.799108e-16$
Ω_{-1}	$2.175217e+00$	Ω_{18}	$-3.640745e-16$
	Ω_0	$-3.493238e+00$	

TABLE III: 2-term Connection Coefficients with $N = 20$, and $d = 2$

Letting $\tilde{u}$ be an approximate solution of homogeneous second order linear ordinary differential equation (14) of the form (4) and using the wavelet basis of level N and resolution j for such approximation then

$$\tilde{u}(x) = \sum_{k=1-N}^{2^j} c_k 2^{j/2}\varphi(2^j x - k) = \sum_{k=1-N}^{2^j} c_k \varphi_{j,k}(x), \quad (15)$$

where the c_k coefficients are unknown. Therefore, letting $\tilde{u} = u$ and putting this last expression into (14) then

$$\frac{d^2}{dx^2}\left(\sum_{k=1-N}^{2^j} c_k\varphi_{j,k}(x)\right) + p_1\frac{d}{dx}\left(\sum_{k=1-N}^{2^j} c_k\varphi_{j,k}(x)\right) + p_2\sum_{k=1-N}^{2^j} c_k\varphi_{j,k}(x) = 0 \quad (16)$$

$$\sum_{k=1-N}^{2^j} c_k 2^{2j}\varphi''_{j,k}(x) + p_1\sum_{k=1-N}^{2^j} c_k 2^j\varphi'_{j,k}(x) + p_2\sum_{k=1-N}^{2^j}\varphi_{j,k}(x) = 0$$

Multiplying this last expression by $\varphi_{j,n}(x)$ and taking inner product on both sides then

$$\sum_{k=1-N}^{2^j} c_k 2^{2j}\int_{-\infty}^{\infty}\varphi_{j,n}\varphi''_{j,k}dx + p_1\sum_{k=1-N}^{2^j} c_k 2^j\int_{-\infty}^{\infty}\varphi_{j,n}\varphi'_{j,k}dx + p_2\sum_{k=1-N}^{2^j} c_k\int_{-\infty}^{\infty}\varphi_{j,n}\varphi_{j,k}dx = 0 \quad (17)$$

Since $supp(\varphi) = [0, N-1]$ the integral turns into

$$\int_{-\infty}^{\infty}\varphi_{j,n}\varphi^d_{j,k}dx = \underbrace{\int_{-\infty}^{0}\varphi_{j,n}\varphi^d_{j,k}dx}_{\to 0} + \int_{0}^{N-1}\varphi_{j,n}\varphi^d_{j,k}dx + \underbrace{\int_{N-1}^{\infty}\varphi_{j,n}\varphi^d_{j,k}dx}_{\to 0} = \Omega^{0,d}_{k,n} := \Omega^{0,d}_{0,k-n} \quad (18)$$

Define $\delta_{k,n}(x)$ as $\delta_{k,n}(x) = \int_{-\infty}^{\infty}\varphi_{j,k}\varphi_{j,n}dx = \int_0^{N-1}\varphi_{j,k}\varphi_{j,n}dx$ as the *Kronecker* delta. This way we arrive at

$$\sum_{k=1-N}^{2^j} c_k 2^{2j}\int_0^{N-1}\varphi''_{j,k}\varphi_{j,n}dx + p_1\sum_{k=1-N}^{2^j} c_k 2^j\int_0^{N-1}\varphi_{j,n}\varphi'_{j,k}dx + p_2\sum_{k=1-N}^{2^j} c_k\int_0^{N-1}\varphi_{j,n}\varphi_{j,k}dx = 0 \Rightarrow$$

$$\sum_{k=1-N}^{2^j} c_k\Omega^{0,2}_{0,k-n} + p_1\sum_{k=1-N}^{2^j} c_k\Omega^{0,1}_{0,k-n} + p_2\sum_{k=1-N}^{2^j} c_k\delta_{k,n} = 0 \quad (19)$$

Making use of the boundary conditions $u(0) = a$ and $u(1) = b$ under (15) then

$$\begin{aligned} u(0) &= \sum_{k=1-N}^{2^j} c_k 2^{j/2}\varphi(-k) = a \quad \to \sum_{k=1-N}^{2^j} c_k 2^{j/2}\delta_{k,n}(0) = a \\ u(1) &= \sum_{k=1-N}^{2^j} c_k 2^{j/2}\varphi(2^j - k) = b \to \sum_{k=1-N}^{2^j} c_k 2^{j/2}\delta_{k,n}(1) = b \end{aligned} \quad (20)$$

Notice that for each k, n with $k - n = -N+2, -N+3, \ldots, 0, \ldots, N-3, N-2$ we can build a linear system $TC = B$ where T is a squared $2N-3$ diagonal matrix of size $2^j + (N-1)$, whose elements are the values of $\Lambda^{0,2}$ (2-term connection coefficients) along with the parameters p_1 and p_2 depending on k and n, $C^T = [c_{1-N}, c_{2-N}, \ldots, c_{2^j-1}, c_{2^j}]$ and B is a vector with almost all of its elements as zero but those corresponding to the boundary conditions, i.e., $B^T = [a, \underbrace{\mathbf{0}}_{1 \text{ x } 2^j+(N-3)}, b]$.

It is clear that the wavelet resolution determines how many points will be deployed in the approximation. This way, the interval $[0,1]$ of length 1 will be partitioned by sub-intervals of length $1/2^j$. Now, as j is incremented then

$$\lim_{j\to\infty}\left(1 - \tfrac{1}{2^j}\right) = 1 - \underbrace{\lim_{j\to\infty}\left(\tfrac{1}{2^j}\right)}_{\to 0} = 1$$

Therefore we need at least the parameter “j” to be 7 to obtain an adequate representation of the exact solution for Equation 14, which relies on the equation parameters p_1 and p_2. Such closed-form solution hangs on whether the discriminant of the corresponding characteristic equation $r^2 + p_1 r + p_2 = 0$ is positive, negative or zero, i.e, $p_1^2 - 4p_2 > 0, < 0, = 0$.

As an example of the *Wavelet-Galerkin* method develop so far we perform the simplest case $j = 0$ and $N = 6$, in which instance the system $TC = B$ follows the below handmade structure, where the first part of (20) has been used to introduce the first and last row of T.

Setting $j = 0$ and $N = 6$, we seek for an approximation of the form

$$u = \sum_{k=1-6}^{2^0} c_k 2^{0/2}\varphi(2^0 x - k) = \sum_{k=-5}^{1} c_k\varphi(x-k) \quad (21)$$

and therefore for T, C and B satisfying the mentioned conditions concerning dimensionality and building. In this fashion we get

$$T = \left[\begin{matrix} 0 & \varphi(4) \\ \Omega_1^{0,2} + p_1\Omega_1^{0,1} & \Omega_0^{0,2} + p_1\Omega_0^{0,1} + p_2 \\ \Omega_2^{0,2} + p_1\Omega_2^{0,1} & \Omega_1^{0,2} + p_1\Omega_1^{0,1} \\ \Omega_3 + p_1\Omega_3^{0,1} & \Omega_2^{0,2} + p_1\Omega_2^{0,1} \\ \Omega_4 + p_1\Omega_4^{0,1} & \Omega_3^{0,2} + p_1\Omega_3^{0,1} \\ \Omega_5^{0,2} + p_1\Omega_5^{0,1} & \Omega_4^{0,2} + p_1\Omega_4^{0,1} \\ 0 & 0 \end{matrix}\right. \quad (22)$$

$$\begin{matrix} \varphi(3) & \varphi(2) \\ \Omega_{-1}^{0,2} + p_1\Omega_{-1}^{0,1}\Omega_{-5}^{0,2} + p_1\Omega_{-5}^{0,1} & \Omega_{-2}^{0,2} + p_1\Omega_{-2}^{0,1} \\ \Omega_0^{0,2} + p_1\Omega_0^{0,1} + p_2\Omega_{-4} + p_1\Omega_{-4}^{0,1} & \Omega_{-1}^{0,2} + p_1\Omega_{-1}^{0,1} \\ \Omega_1^{0,2} + p_1\Omega_1^{0,1}\Omega_{-3} + p_1\Omega_{-3}^{0,1} & \Omega_0^{0,2} + p_1\Omega_0^{0,1} + p_2 \\ \Omega_2^{0,2} + p_1\Omega_2^{0,1}\Omega_{-2}^{0,2} + p_1\Omega_{-2}^{0,1} & \Omega_1^{0,2} + p_1\Omega_1^{0,1} \\ \Omega_3^{0,2} + p_1\Omega_3^{0,1}\Omega_{-1}^{0,2} + p_1\Omega_{-1}^{0,1} & \Omega_2^{0,2} + p_1\Omega_2^{0,1} \\ \varphi(4) & \varphi(3) \end{matrix}$$

$$\left.\begin{matrix} \varphi(1) & 0 & 0 \\ \Omega_{-3}^{0,2} + p_1\Omega_{-3}^{0,1} & \Omega_{-4}^{0,2} + p_1\Omega_{-4}^{0,1} & \Omega_{-5}^{0,2} + p_1\Omega_{-5}^{0,1} \\ \Omega_{-2}^{0,2} + p_1\Omega_{-2}^{0,1} & \Omega_{-3}^{0,2} + p_1\Omega_{-3}^{0,1} & \Omega_{-4} + p_1\Omega_{-4}^{0,1} \\ \Omega_{-1}^{0,2} + p_1\Omega_{-1}^{0,1} & \Omega_{-2}^{0,2} + p_1\Omega_{-2}^{0,1} & \Omega_{-3} + p_1\Omega_{-3}^{0,1} \\ \Omega_0^{0,2} + p_1\Omega_0^{0,1} + p_2 & \Omega_{-1}^{0,2} + p_1\Omega_{-1}^{0,1} & \Omega_{-2}^{0,2} + p_1\Omega_{-2}^{0,1} \\ \Omega_1^{0,2} + p_1\Omega_1^{0,1} & \Omega_0^{0,2} + p_1\Omega_0^{0,1} + p_2 & \Omega_{-1}^{0,2} + p_1\Omega_{-1}^{0,1} \\ \varphi(2) & \varphi(1) & 0 \end{matrix}\right]$$

with $C^T = [c_{-5} \quad c_{-4} \quad c_{-3} \quad c_{-2} \quad c_{-1} \quad c_0 \quad c_1]$ and $B^T = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$. Solving $C = T^{-1}B$ we get

$$C = \begin{bmatrix} -0.9972 \\ -0.8776 \\ 0.1279 \\ 1.0543 \\ 1.0870 \\ 0.2479 \\ -0.5059 \end{bmatrix} \quad (23)$$

This particular case has the exact solution $u(x) = \cos(x) - \cot(1)\sin(x)$, hence the absolute error can be computed. We showthe the results in Figure 4.

Nevertheless, for more complex cases the matrices 22 and 23 cannot be displayed since their sizes is considerably large. Besides, it is reasonable to think that they might not be well conditioned for several instances; in such cases, the parameters should be changed to achieve more reliable solutions.

Next, further results obtained via computational computation after fixing “j” as 8, letting $N = 6$ are presented in Figures 5, 6, **??** and 9. Several runs are executed with different *Dirichlet* conditions and $p-$coefficients in order to explore the method capability.

The studied cases applying the method are the following. These instances are the most representative since for case 1, 3 and 2, and 4 the discriminant satisfies $> 0, = 0, < 0$ correspondingly.

Case	p_1	p_2	a	b
1	1	-1	2	-1
2	0	$(9.5\pi)^2$	2	-1
3	5	6	1	1
4	3	25	1	1

TABLE IV: Computated cases

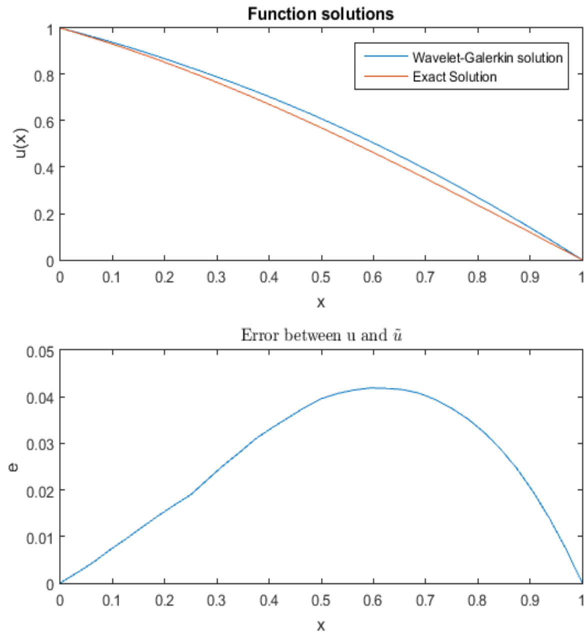

Fig. 4: Scaling and Wavelet functions for DN6

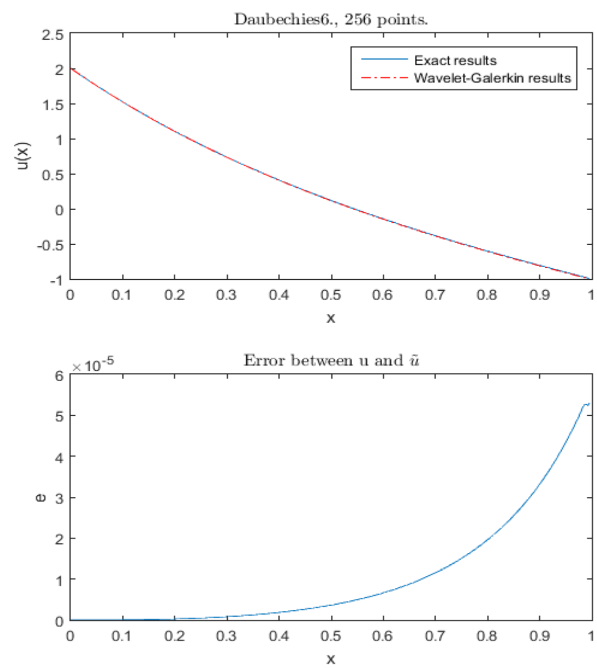

Fig. 5: Method results for case 1

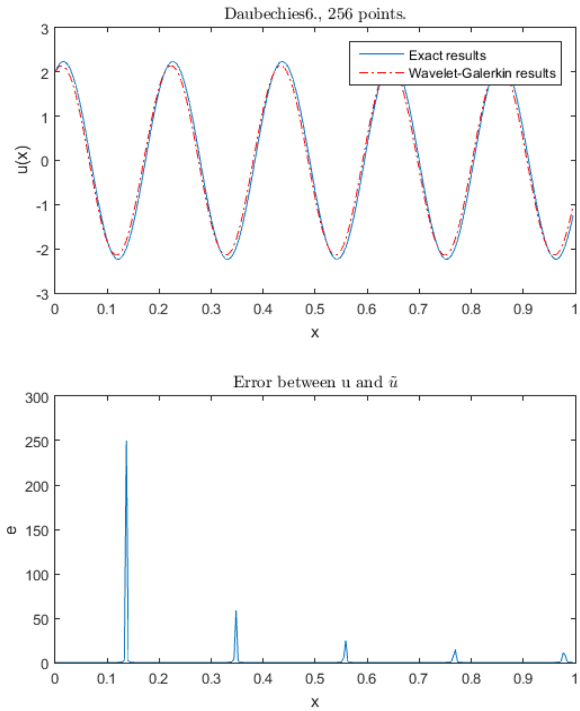

Fig. 6: Method results for case 2

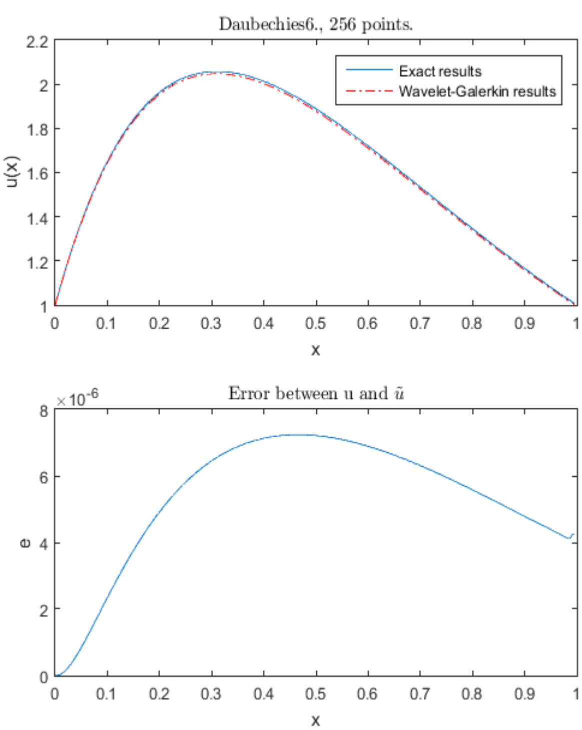

Fig. 7: Method results for case 3

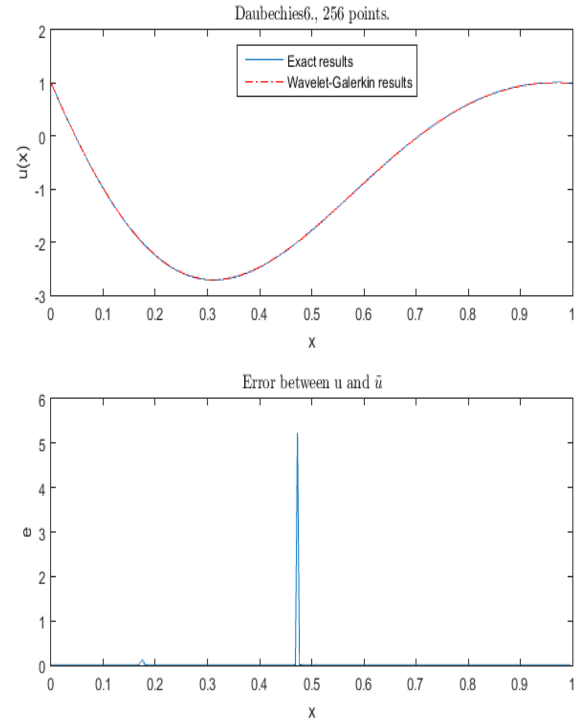

Fig. 8: Method results for case 4

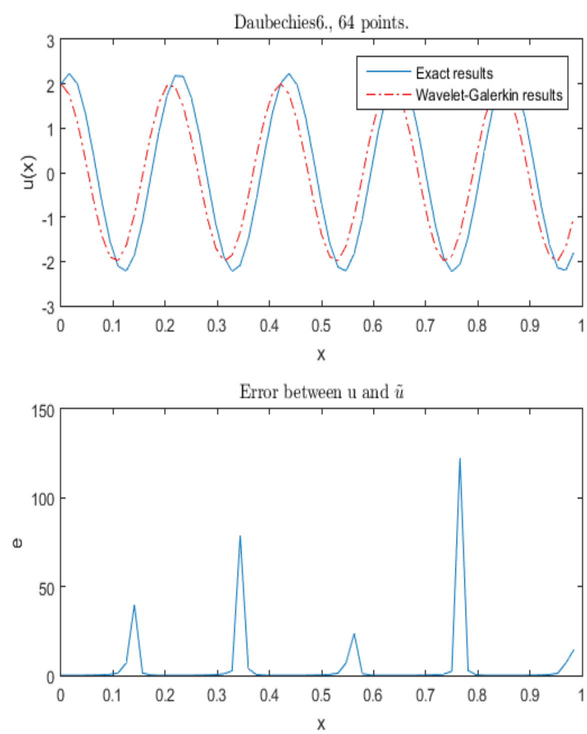

Fig. 9: Method results for case 2 with $j = 6$

In general the approximate solutions are well posed and their behavior or shape is similar to the exact one. On one hand, for instance, for the case results displayed in Figures 5 and 6 we see that the error has a scale of $*10^{-5}$, hence providing a good approximation for the differential equation. On the other hand, regarding the non-positive discriminant, the errors between the approximate and exact solutions present a decreasing error for every minimal (valley). Such consequence of the method performance is better observed in Figures 7 and 8, where the sinusoidal component is softer and hence the error smaller.

Now, for the case 3, we see how the deviation takes the exact solution form in the temporal interval. As we modify the wavelet resolution it is expected to get worse approximations with greater error and possible shifts, as seen in Figure 9, where the resolution parameter was settled to 6. Nevertheless, possible fluctuations over the wavelet level N will not improve the method results since less points are considered, but enhancing the approximations in these points for differential equations with non-polynomial independent term f.

IV. Concluding remarks

The *Wavelet-Galerkin* method was applied on the homogeneous second order ordinary differential equation with constant coefficients with *Dirichlet* boundary conditions, getting reliable results with low absolute error. The variations effect on the method parameters was studied for that particular case, making the proper distinctions between them. Although any convergence or stability were introduced in this work, the low quantity of parameters, as well as their explanation, let set a pair of boundedness conditions concerning the parameters. About the consistency aspect of the method we can say that such condition is satisfied since the method seems not to have any contradictions. In general, the method also provides good approximations with simple forms for the L differential operator; nonetheless, using the methodology here introduced and developed it is possible to computate more complex cases and even consider independents terms that can be represented using series expansion and taking advantage of the moments function M (13).

Finally, we can introduce *Neumann*, mixed or even *Robin* boundary conditions in the problem (1) and exploit the method adequacy. Additionally, we are able to computate the connection coefficients for greater values of d_1 and d_2, letting solve non-linear combinations of the unknown function in the initial value problem. Despite of this, a further study of the conditionality of T is possible in (12) with regard to the system solution related to these coefficients.

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