Application of the Wavelet-Galerkin Method on the homogeneous second order linear ordinary differential equation with constant coefficients

Research practise 2 project presentation

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Precedings

Those who developed the theory

Figure 1: Numerical Methods for ODE and Wavelets Timeline

Precedings

Those who developed the theory

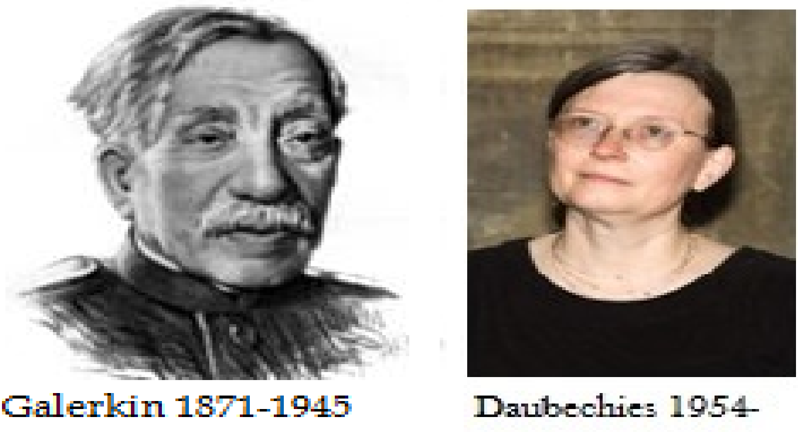


Figure 2: Wavelet-Galerkin Method Timeline

Mathematical formulation

Consider ϕ_i as a base of $L^2([0,1])$ and every ϕ_i satisfying C^2 on $[0,1]$ such that $\phi_i(0)=a, \phi_i(1)=b$, u_0 as an approximate solution of the equation with Λ, S as a finite set of indices i and the subspace $span\{\phi_i : i \in \Lambda\}$ respectively so that [3]:

$$\langle\, Lu_0 - f, \phi_i \rangle = 0, \quad \forall i \in \Lambda \tag{1}$$

$$u_0 = \sum_{k\in\Lambda} a_k \phi_k \in S. \tag{2}$$

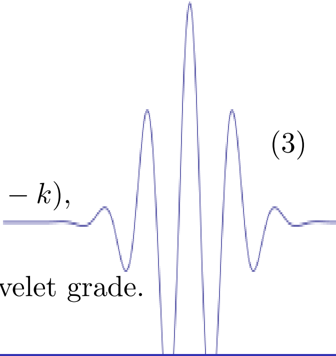
Letting $\tilde{u}$ of the form (2) as the approximate solution of (1) it is intended that the residue $R = L\tilde{u} - f$ to be orthogonal to the chosen base on $\mathcal{D}_L$.

Formulations

Problem: Wavelet-Galerkin method

The Wavelet-Galerkin method considers $\phi(x) = \Psi_{j,k}(x) = 2^{j/2}\Psi(2^j x - k)$ as a wavelet basis for $L^2([0,1])$ satisfying the boundary conditions $\Psi_{j,k}(0) = \Psi_{j,k}(1) = 0$ and $\forall j, k \in \Lambda$ then $\Psi_{j,k}$ is C^2. Using the Daubechies [1] wavelets then both $\varphi_{j,k}$ and $\Psi_{j,k}$ can be computed setting the scaling and mother wavelet functions respectively as

$$\begin{aligned} \varphi(x) &= \sum_{k=0}^{L-1} a_k \varphi(2x-k) \\ \Psi(x) &= \sum_{k=2-L}^{L} (-1)^k a_{1-k} \varphi(2x-k), \end{aligned} \tag{3}$$



with a_k characterized for the N-Daubechies wavelet grade.

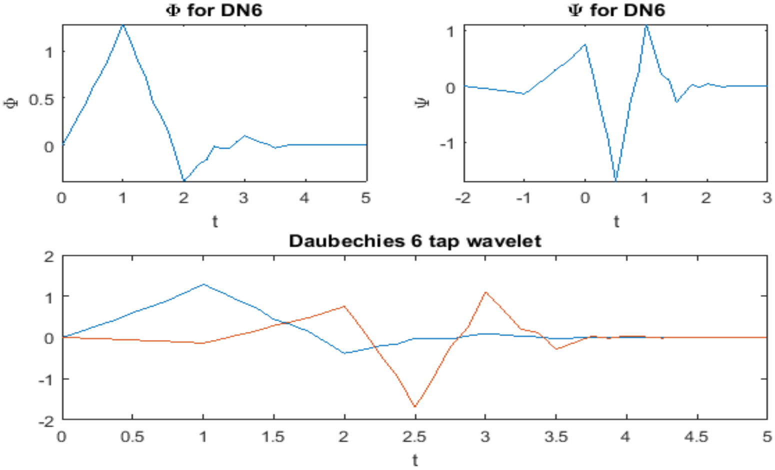


Figure 3: Scaling and Wavelet functions for DN6

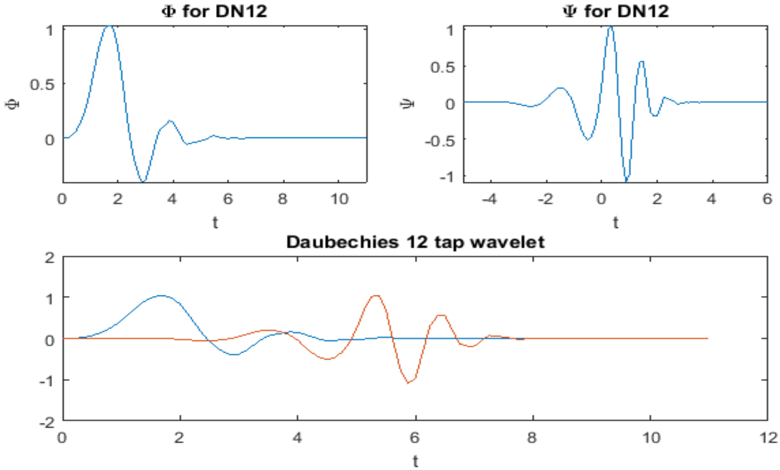


Figure 4: Scaling and Wavelet functions for DN12

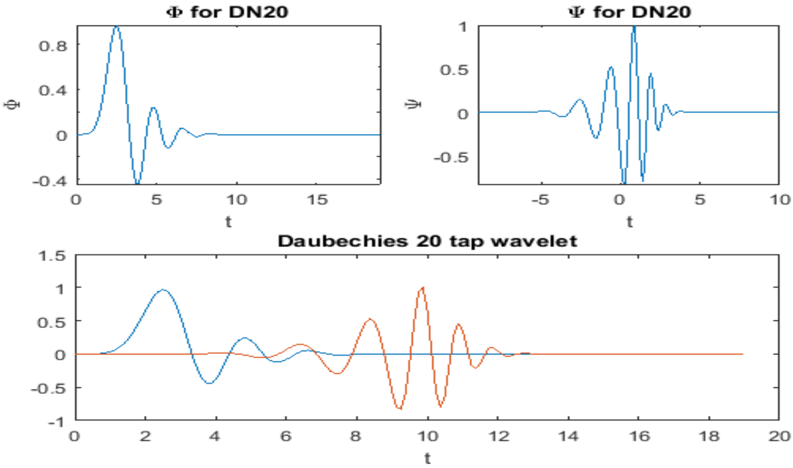


Figure 5: Scaling and Wavelet functions for DN20

Formulations

Connection Coefficients

It is neccesary to computate several expressions in order to find the solution of differential equation by using this method, specifically the *Connection coefficients* [2] defined as follows:

Connection coefficients

$$\Omega_{j,k}^{m,n}(x) = \int_{-\infty}^{\infty} \varphi^{(m)}(y-j)\varphi^{(n)}(y-k)\mathrm{d}y. \tag{4}$$

Formulations

2-term Connection Coefficients

Taking the respective derivatives and simplificating the following system of linear equations is found, where $\Omega^{m,n}$ is the unknown vector to be calculated.

$$\begin{pmatrix} T - \frac{1}{2^{d-1}} I \\ M^d \end{pmatrix} \Omega^{m,n} = \begin{pmatrix} 0 \\ d! \end{pmatrix} \tag{5}$$

where $d = m + n$, $T = \sum_i a_i a_{q-2l+i}$ and M_i^k are the moments of φ_i defined as

$$M_i^k = \int_{-\infty}^{\infty} x^k \varphi_i(x) dx,$$

satisfying $M_0^0 = 1$.

$\Lambda^{0,2}$	$N=6,\ j=0$	$N=6,\ j=7$
Ω_{-4}	$5.357142857141725e-03$	$8.777142857143009e+01$
Ω_{-3}	$1.142857142857160e-01$	$1.872457142857140e+03$
Ω_{2}	$-8.761904761904885e-01$	$-1.435550476190474e+04$
Ω_{-1}	$3.390476190476218e+00$	$5.554956190476182e+04$
Ω_{0}	$-5.267857142857142e+00$	$-8.630857142857110e+04$
Ω_{1}	$3.390476190476168e+00$	$5.554956190476169e+04$
Ω_{2}	$-8.761904761904653e-01$	$-1.435550476190469e+04$
Ω_{3}	$1.142857142857138e-01$	$1.872457142857137e+03$
Ω_{4}	$5.357142857143558e-03$	$8.777142857143159e+01$

Table 1: 2-term Connection Coefficients holding $N=6$, and $d=2$

$\Lambda^{0,2}$	$N=12,\ j=0$	$\Lambda^{0,2}$	$N=6,\ j=4$
Ω_{-18}	$3.928343e-15$	Ω_1	$2.175217e+00$
Ω_{-17}	$-3.486099e-16$	Ω_2	$-6.066894e-01$
Ω_{-16}	$2.858395e-15$	Ω_3	$2.546974e-01$
Ω_{-15}	$-2.399663e-13$	Ω_4	$-1.054297e-01$
Ω_{-14}	$-5.015915e-11$	Ω_5	$3.758004e-02$
Ω_{-13}	$-2.219929e-09$	Ω_6	$-1.078072e-02$
Ω_{-12}	$6.114256e-09$	Ω_7	$2.357271e-03$
Ω_{-11}	$1.222971e-07$	Ω_8	$-3.693880e-04$
Ω_{-10}	$-2.579303e-06$	Ω_9	$3.852452e-05$

Ω_{-9}	$3.852452e-05$	Ω_{10}	$-2.579303e-06$
Ω_{-8}	$-3.693880e-04$	Ω_{11}	$1.222971e-07$
Ω_{-7}	$2.357271e-03$	Ω_{12}	$6.114256e-09$
Ω_{-6}	$-1.078072e-02$	Ω_{13}	$-2.219929e-09$
Ω_{-5}	$3.758004e-02$	Ω_{14}	$-5.015874e-11$
Ω_{-4}	$-1.054297e-01$	Ω_{15}	$-2.399884e-13$
Ω_{-3}	$2.546974e-01$	Ω_{16}	$2.821739e-15$
Ω_{-2}	$-6.066894e-01$	Ω_{17}	$-3.799108e-16$
Ω_{-1}	$2.175217e+00$	Ω_{18}	$-3.640745e-16$
Ω_{0}	$-3.493238e+00$		

Table 2: 2-term Connection Coefficients with $N = 20$, and $d = 2$

Formulations

2-term Connection Coefficients

Let us consider the general integral-differential equation depending on u with $x \in [a, b]$:

$$f(x, \tfrac{du}{dx}, \tfrac{d^2u}{dx^2}, \ldots, \int^x u dx_1, \int^x \int^{x_1} u dx_2 dx_1, \ldots) = 0. \tag{6}$$

Following the common notation for the approximation of u according to (2), we have $\tilde{u}$ is as follows:

$$\tilde{u}(x) = \sum_{k=1-L}^{2^j} c_k \varphi_{j,k}(x) = \sum_{k=1-L}^{2^j} c_k 2^{j/2} \varphi(2^j x - k). \tag{7}$$

Formulations

2-term Connection Coefficients

Using this approximation, the coefficients c_k are determined by applying the inner product and solving (8) for $k = 1 - L, \ldots, 2^j$.

$$\int_a^b \varphi_{j,k}(x) f(x, \tfrac{d\tilde{u}}{dx}, \tfrac{d^2\tilde{u}}{dx^2}, \ldots, \int^x \tilde{u} dx_1, \int^x \int^{x_1} \tilde{u} dx_2 dx_1, \ldots) = 0 \qquad (8)$$

Consider the problem

Second order linear ordinary differential equation

$$\frac{d^2u}{dx^2} + \alpha \frac{du}{dx} + \beta u = 0, \quad 0 < x < 1,$$
$$u(0) = a \quad \text{and} \quad u(1) = b.$$

Using the wavelet basis of level N and resolution j for the approximation, where the c_k coefficients are unknown, then

$$\tilde{u}(x) = \sum_{k=1-N}^{2^j} c_k 2^{j/2} \varphi(2^j x - k) = \sum_{k=1-N}^{2^j} c_k \varphi_{j,k}(x),$$

Application

General example

Specification

$$\sum_{k=1-N}^{2^j} c_k \Omega^{0,2}_{0,k-n} + \alpha \sum_{k=1-N}^{2^j} c_k \Omega^{0,1}_{0,k-n} + \beta \sum_{k=1-N}^{2^j} c_k \delta_{k,n} = 0$$

where

$$\delta_{k,n}(x) = \int_{-\infty}^{\infty} \varphi_{j,k}\varphi_{j,n} dx = \int_{0}^{N-1} \varphi_{j,k}\varphi_{j,n} dx,$$

$$\Omega^{0,2}_{0,k-n} = 2^{2j} \int_{-\infty}^{\infty} \varphi''_{j,k}\varphi_{j,n} dx = 2^{2j} \int_{0}^{N-1} \varphi''_{j,k}\varphi_{j,n} dx,$$

$$\Omega^{0,1}_{0,k-n} = 2^{j} \int_{-\infty}^{\infty} \varphi_{j,n}\varphi'_{j,k} dx = 2^{j} \int_{0}^{N-1} \varphi_{j,n}\varphi'_{j,k} dx$$

Boundary conditions

$$u(0) = \sum_{k=1-N}^{2^j} c_k 2^{j/2} \varphi(-k) = a \quad \rightarrow \sum_{k=1-N}^{2^j} c_k 2^{j/2} \delta_{k,n}(0) = a$$

$$u(1) = \sum_{k=1-N}^{2^j} c_k 2^{j/2} \varphi(2^j - k) = b \rightarrow \sum_{k=1-N}^{2^j} c_k 2^{j/2} \delta_{k,n}(1) = b$$

Setting $j = 0$ and $N = 6$ we seek for an approximation of the form

$$u = \sum_{k=1-6}^{2^0} c_k 2^{0/2} \varphi(2^0 x - k) = \sum_{k=-5}^{1} c_k \varphi(x - k)$$

Hence $TC = B$ with

$$C^T = [c_{-5} \quad c_{-4} \quad c_{-3} \quad c_{-2} \quad c_{-1} \quad c_0 \quad c_1],$$
$$B^T = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] \quad \text{and}$$

Application

Solve $TC = B$

$$T = \left[\begin{matrix} 0 & \varphi(4) & \varphi(3) \\ \Omega_1^{0,2} + p_1\Omega_1^{0,1} & \Omega_0^{0,2} + p_1\Omega_0^{0,1} + p_2 & \Omega_{-1}^{0,2} + p_1\Omega_{-1}^{0,1}\Omega_{-5}^{0,2} + p_1\Omega_{-5}^{0,1} \\ \Omega_2^{0,2} + p_1\Omega_2^{0,1} & \Omega_1^{0,2} + p_1\Omega_1^{0,1} & \Omega_0^{0,2} + p_1\Omega_0^{0,1} + p_2\Omega_{-4} + p_1\Omega_{-4}^{0,1} \\ \Omega_3 + p_1\Omega_3^{0,1} & \Omega_2^{0,2} + p_1\Omega_2^{0,1} & \Omega_1^{0,2} + p_1\Omega_1^{0,1}\Omega_{-3} + p_1\Omega_{-3}^{0,1} \\ \Omega_4 + p_1\Omega_4^{0,1} & \Omega_3^{0,2} + p_1\Omega_3^{0,1} & \Omega_2^{0,2} + p_1\Omega_2^{0,1}\Omega_{-2}^{0,2} + p_1\Omega_{-2}^{0,1} \\ \Omega_5^{0,2} + p_1\Omega_5^{0,1} & \Omega_4^{0,2} + p_1\Omega_4^{0,1} & \Omega_3^{0,2} + p_1\Omega_3^{0,1}\Omega_{-1}^{0,2} + p_1\Omega_{-1}^{0,1} \\ 0 & 0 & \varphi(4) \end{matrix} \right.$$

$$\left. \begin{matrix} \varphi(2) & \varphi(1) & 0 & 0 \\ \Omega_{-2}^{0,2} + p_1\Omega_{-2}^{0,1} & \Omega_{-3}^{0,2} + p_1\Omega_{-3}^{0,1} & \Omega_{-4}^{0,2} + p_1\Omega_{-4}^{0,1} & \Omega_{-5}^{0,2} + p_1\Omega_{-5}^{0,1} \\ \Omega_{-1}^{0,2} + p_1\Omega_{-1}^{0,1} & \Omega_{-2}^{0,2} + p_1\Omega_{-2}^{0,1} & \Omega_{-3}^{0,2} + p_1\Omega_{-3}^{0,1} & \Omega_{-4} + p_1\Omega_{-4}^{0,1} \\ \Omega_0^{0,2} + p_1\Omega_0^{0,1} + p_2 & \Omega_{-1}^{0,2} + p_1\Omega_{-1}^{0,1} & \Omega_{-2}^{0,2} + p_1\Omega_{-2}^{0,1} & \Omega_{-3} + p_1\Omega_{-3}^{0,1} \\ \Omega_1^{0,2} + p_1\Omega_1^{0,1} & \Omega_0^{0,2} + p_1\Omega_0^{0,1} + p_2 & \Omega_{-1}^{0,2} + p_1\Omega_{-1}^{0,1} & \Omega_{-2}^{0,2} + p_1\Omega_{-2}^{0,1} \\ \Omega_2^{0,2} + p_1\Omega_2^{0,1} & \Omega_1^{0,2} + p_1\Omega_1^{0,1} & \Omega_0^{0,2} + p_1\Omega_0^{0,1} + p_2 & \Omega_{-1}^{0,2} + p_1\Omega_{-1}^{0,1} \\ \varphi(3) & \varphi(2) & \varphi(1) & 0 \end{matrix} \right]$$

Solving the last system we find

$$C^T = [-0.9972 \quad -0.8776 \quad 0.1279 \quad 1.0543 \quad 1.0870 \quad 0.2479 \quad -0.5059]$$

and therefore

$$\tilde{u}(x) = \sum_{k=-5}^{1} c_k \varphi(x-k) = -0.9972\varphi(x+5) - 0.8776\varphi(x+4) + \ldots$$
$$0.1279\varphi(x+3) + 1.0543\varphi(x+2) + 1.0870\varphi(x+1) + \ldots$$
$$0.2476\varphi(x) - 0.5059\varphi(x-1).$$

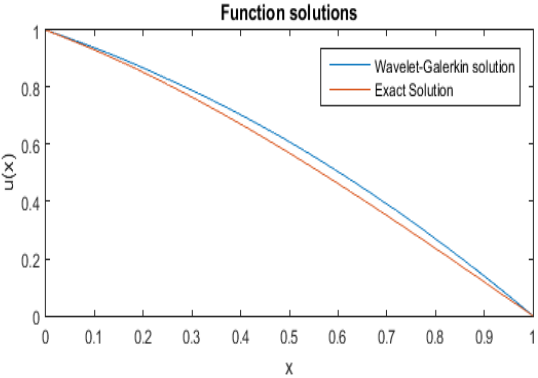


Figure 6: Exact and approximate solution of $u'' + u = 0$, $u(0) = 1$, $u(1) = 0$.

Results

Absolute error

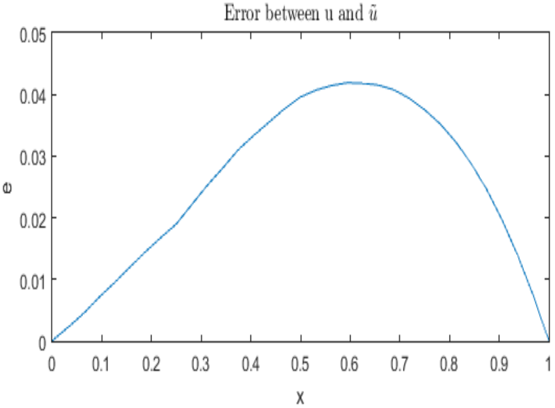


Figure 7: Error between exact and approximate solutions

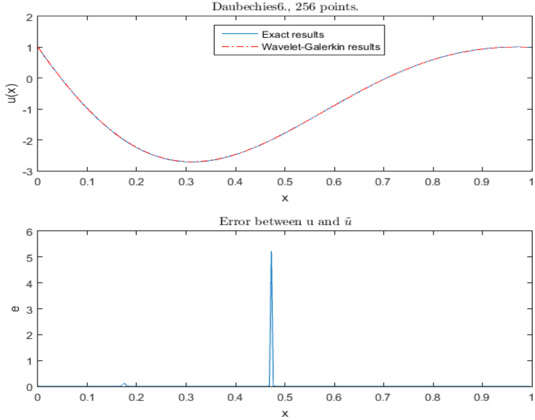


Figure 8: Exact and approximate solution of $u^{''} + 3u^{'} + 25u = 0$, $u(0) = 1$, $u(1) = 1$.

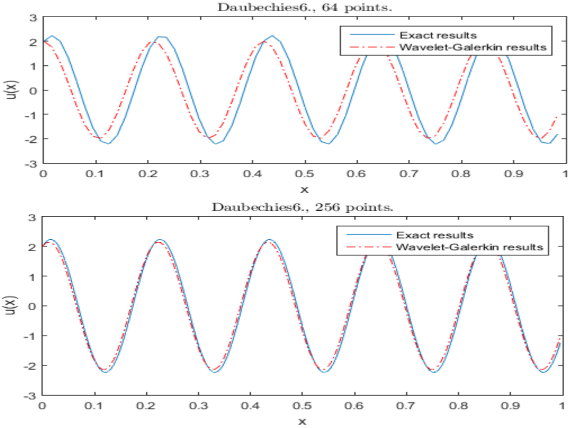


Figure 9: Exact and approximate solution of $u'' + (9.5\pi)^2 u = 0$, $u(0) = 2$, $u(1) = -1$.

Acknowledgment

THANK YOU FOR YOUR ATTENTION!

QUESTIONS?

Bibliography

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