IMPLEMENTATION OF FINITE ELEMENTS METHOD ON A DIFFUSION-ADVECTION PROBLEM

Research practise 1 progress presentation

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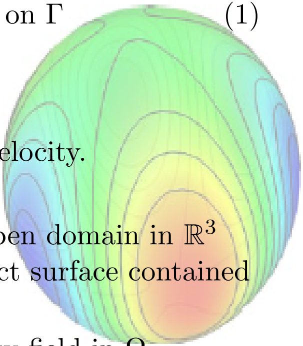
Introduction

Main equations describing the phenomena

Time dependent equation on steady surfaces

$$c_t + \mathbf{w} \cdot \nabla_\Gamma c - D \triangle_\Gamma c = 0 \quad \text{on } \Gamma \tag{1}$$

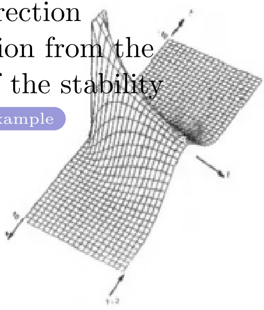


- $c \Rightarrow$ chemical species concentration.
- $\mathbf{u} \Rightarrow$ the *Darcy* or chemical crossflow velocity.
- $\mathbf{D} \Rightarrow$ the diffusion coefficient.
- $\Omega \Rightarrow$ the physical domain seen as an open domain in $\mathbb{R}^3$ and therefore Γ a connected C^2 compact surface contained in Ω.
- $\mathbf{w} : \Omega \to \mathbb{R}^3 \Rightarrow$ a divergence-free velocity field in Ω.
- $\triangle_\Gamma \Rightarrow$ *Laplace-Beltrami* operator on Γ.

Time dependent equation in space

$$\frac{\partial C}{\partial t} + \nabla \cdot (\mathbf{u}C) = D\nabla^2 C \qquad (2)$$

In this case $\mathbf{u}$ can be considered as the wind velocity and it is usually taken as constant in any horizontal direction $\vec{\mathbf{u}} = (U, V, 0)$. So it is possible to build a solution from the Finite Difference Method as well as a study of the stability using the *von Neumann* stability criterion. Example



Von Neumann criterion

The difference method for an initial value problem (for a differential equation with constant coefficients) with a bounded solution is stable if every solution to the finite difference equation having the form $c_j^n = \xi^n e^{i\beta j}$, ($\beta$ real, $\xi = \xi(\beta)$ complex) has the property $\mid \xi \mid \leq 1$. Example

Finite elements method first criterion idea

In the study of the elastic theory where three kinds of magnitudes, stresses, strains and displacements determine the solution by using the finite elements method, if certain conditions concerning completeness and the good behavior of the approximate solution are satisfied then convergence is insured. Continuation

Developments

Example proposition

Solve the advection diffusion problem [1] governed by (2) subject to:

Problem conditions

$$\vec{u} = (U, V, 0) = (-\sin\theta, \cos\theta, 0)$$

$$c_0(x, y) = \begin{cases} 50(1 + \cos\frac{\pi R}{4}) & , \text{if } R < 4 \\ 0 & , \text{if } R > 4 \end{cases}$$

$$\text{where} \begin{cases} \theta & = \arctan\frac{y}{x} \\ R^2 & = (x - x_0)^2 + (y - y_0)^2 \\ (x_0, y_0) & = (5, -10) \end{cases}.$$

Developments

Example results

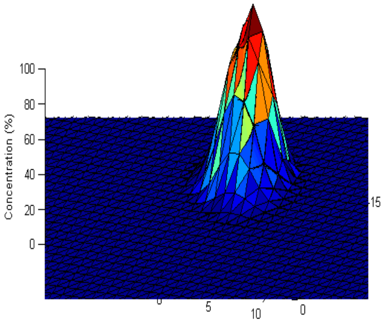


Figure 1: Concentration percentage evolution for 5 and 25 segs

Developments

Example results

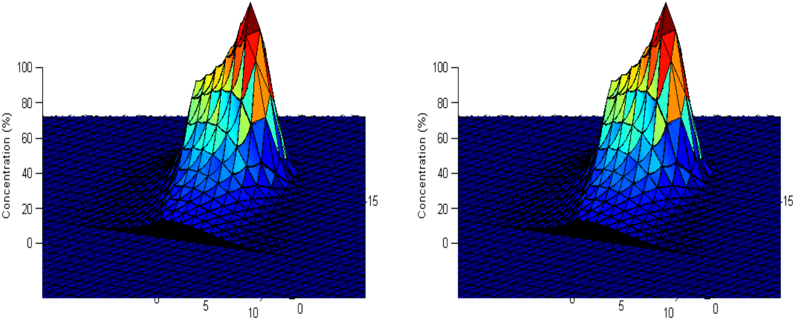


Figure 2: Concentration percentage evolution for 60 and 150 segs

Developments

Example results

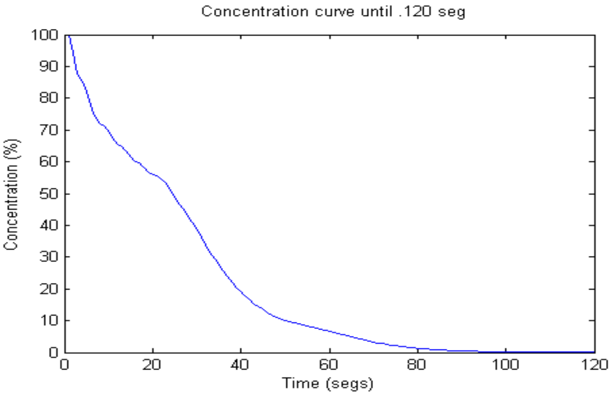


Figure 3: Concentration percentage curve until 120 segs

Continuation

Taking $\vec{u}$ as $\vec{u} = (U, 0, 0)$ and using the $vonNeumann$ stability criterion we find that (2) is convergent if the following condition is hold.

$$\frac{\Delta t}{\Delta x^2}(2k + U\Delta x) \leq 1$$

Continuation

Actual work

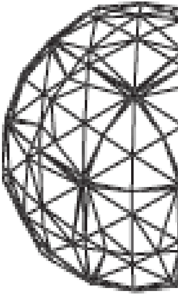
Towards a general formulation

Given the following variation [2] from (1), bilinear form and the functional respectively:

$$\mathbf{w} \cdot \nabla_\Gamma u + c(\mathbf{x})u = f + \epsilon \Delta_\Gamma u \quad \text{on } \Gamma$$

$$a(u, v) := \epsilon \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v \, \mathrm{d}s + \int_\Gamma (\mathbf{w} \cdot \nabla_\Gamma u) v \, \mathrm{d}s + \int_\Gamma uv \, \mathrm{d}s$$

$$f(v) := \int_\Gamma f v \, \mathrm{d}s$$



where $f \in L^2\Gamma$, $c(\mathbf{x}) \geq 0$ and Δ_Γ and ∇_Γ defined as before, find $u \in V$ such that

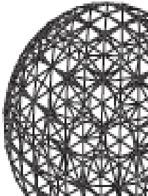
Actual work

$$\boxed{a(u,v) = f(v) \quad , \text{ for all } v \in V}$$

with

$$V = \begin{cases} \left\{ v \in H^1(\Gamma) \mid \int_\Gamma v \, \mathrm{d}s = 0 \right\} & , \text{ if } c = 0 \\ H^1(\Gamma) & , \text{ if } c > 0 \end{cases}$$

where $H^1(\Gamma)$ denotes the Sobolev spaces with $p = 2$ and owing to the *Lax-Milgramm* lemma, there exist a unique solution of this last equation.



Acknowledgment

THANK YOU FOR YOUR ATTENTION!

QUESTIONS?

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[2] OLSHANSKII, MAXIM A., REUSKEN, ARNOLD AND XU, XIANMIN, *A stabilized finite element method for advection–diffusion equations on surfaces*. IMA Journal of Numerical Analysis (2013).