Real Options Valuation for Mining Projects Using a Proposed Numerical Schema Based on Finite Difference Method

J. Mauricio Cuscagua-Lopez Fredy H. Marín-Sanchez

Research Practise 1: Progress Presentation
Mathematical Engineering

October 2, 2015

UNIVERSIDAD EAFIT®

Financial Options Valuation

Let f be the price of an option. To calculate its price the following Partial Differential Equation must be solved [Black and Scholes, 1973]:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} + rs\frac{\partial f}{\partial s} = rf \tag{1}$$

where f is the price of the Option, S is the price of the underlying, σ is the volatility of the underlying and r is the free interest rate risk.

Variable Changes

In search of the calculation for (1) easier, consider the following change of variable.

$$\begin{aligned} H &= e^{r(\tau - t)} f \\ X &= e^{r(\tau - t)} s \end{aligned} \tag{2}$$

Where $H : H(X, t)$ and τ is the maturity time.

Finding the Partial Derivatives I

Deriving (2) we obtain:

$$\frac{\partial f}{\partial t} = e^{-r(\tau-t)} f \frac{\partial H}{\partial t} + re^{-r(\tau-t)} f \tag{3}$$

Applying the Chain Rule we know that:

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial X}\frac{\partial X}{\partial t} + \frac{\partial H}{\partial t}\frac{\partial t}{\partial t} \tag{4}$$

where

$$\frac{\partial X}{\partial t} = -re^{r(\tau-t)} s = -rX \tag{5}$$

Replacing (4) and (5) in (3) we get:

$$\frac{\partial f}{\partial t} = e^{-r(\tau-t)}(-rX\frac{\partial H}{\partial X} + \frac{\partial H}{\partial t}) + rHe^{-r(\tau-t)} \tag{6}$$

Finding the Partial Derivatives II

However, from the expression to H in (2) we have:

$$\frac{\partial f}{\partial s} = e^{-r(\tau - t)} \frac{\partial H}{\partial s} \tag{7}$$

Similar than (4) we know:

$$\frac{\partial H}{\partial s} = \frac{\partial H}{\partial X}\frac{\partial X}{\partial s} + \frac{\partial H}{\partial t}\frac{\partial t}{\partial s} \tag{8}$$

We also know, from deriving (2), that $\frac{\partial X}{\partial s} = e^{r(\tau - t)}$. Using this and (8) in (7) we obtain:

$$\frac{\partial f}{\partial s} = \frac{\partial H}{\partial X} \tag{9}$$

To calculate $\frac{\partial^2 f}{\partial s^2}$ it is enough to derive (9). Thus:

Finding the Partial Derivatives III

$$\frac{\partial^2 H}{\partial s^2} = \frac{\partial}{\partial X}\left(\frac{\partial H}{\partial X}\right) = \frac{\partial}{\partial X}\left(\frac{\partial H}{\partial X}\right)\frac{\partial X}{\partial s} + \frac{\partial}{\partial t}\left(\frac{\partial H}{\partial X}\right)\frac{\partial t}{\partial s} \qquad (10)$$

Finally:

$$\frac{\partial^2 f}{\partial s^2} = e^{r(\tau - t)}\frac{\partial^2 H}{\partial s^2} \qquad (11)$$

The Alternative PDE

Using the results in (6), (9) and (11) in (1) we have:

$$\begin{aligned} & e^{-r(\tau-t)}(-rX\frac{\partial H}{\partial X}+\frac{\partial H}{\partial t}) \\ & +rHe^{-r(\tau-t)}+rXe^{-r(\tau-t)}\frac{\partial H}{\partial X} \\ & +\frac{1}{2}\sigma^2 e^{-2r(\tau-t)}X^2 e^{r(\tau-t)}\frac{\partial^2 H}{\partial X^2}=rHe^{-r(\tau-t)} \end{aligned} \tag{12}$$

Organizing the terms:

$$\frac{\partial H}{\partial t}+\frac{1}{2}\sigma^2 X^2\frac{\partial^2 H}{\partial X^2}=0 \tag{13}$$

Derivatives Discretization

The Finite Difference method sets the following expressions to sample the derivatives [Hull, 2006].

$$
\begin{aligned}
\frac{\partial H}{\partial t} &= \frac{H_{i+1,j} - H_{i,j}}{\Delta t} \\
\frac{\partial H}{\partial X} &= \frac{H_{i+1,j+1} - H_{i+1,j-1}}{2\Delta X} \\
\frac{\partial^2 H}{\partial X^2} &= \frac{H_{i+1,j+1} + H_{i+1,j-1} - 2H_{i+1,j}}{\Delta X^2}
\end{aligned}
\tag{14}
$$

Derivatives Discretization

Using the above expressions in (13), it is obtained the numerical schema

$$H_{i,j} = a_i H_{i+1,j+1} + b_i H_{i+1,j-1} + c_i H_{i+1,j} \tag{15}$$

where $a_i = b_i = \frac{1}{2}\sigma^2 i^2 \Delta t$ and $c_i = (1 - \sigma^2 i^2 \Delta t)$

Note

The coefficients in 15 can be viewed as weights and probability. Note that if $\Delta t \to 0$ then the sum of the weights $\Sigma \to 1$.

Boundary Conditions

For the practice, we'll take an European Call option. Thus:

- If $S = Smax \Rightarrow f = K \Rightarrow H = e^{r(\tau - t)}K$
- If $S = 0 \Rightarrow f = 0 \Rightarrow H = 0$
- If $t = \tau \Rightarrow f = max(S - K, 0) \Rightarrow H = e^{r(\tau - t)}max(S - K, 0)$

Finite Difference Method for Original PDE

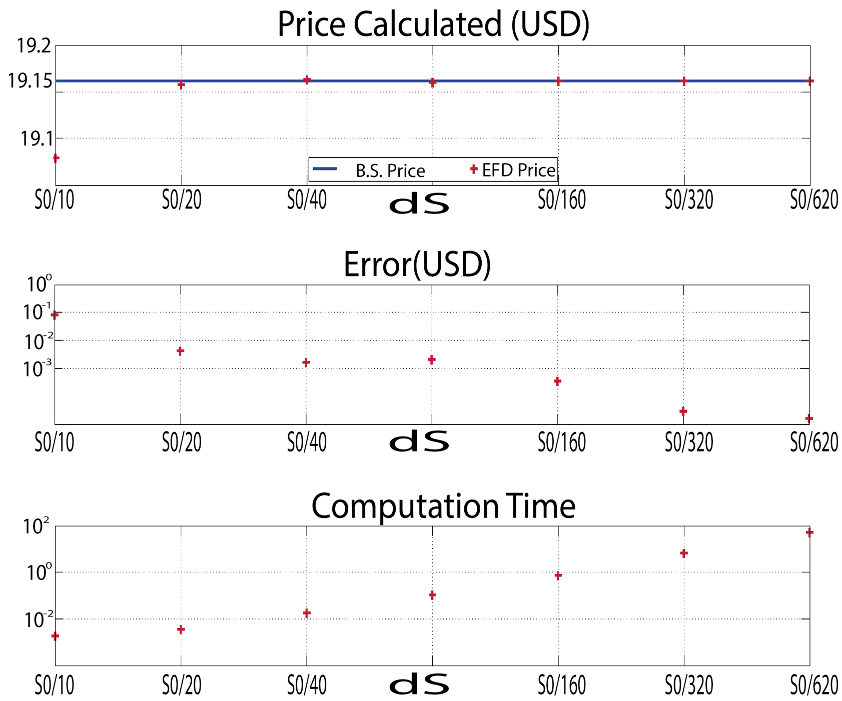

Figure 1: Numerical results for a *call* option with $r = 0.05$, $\sigma = 0.025$ and $k = 100$. Taken from [Velásquez et al., 2015]

Finite Difference Method for the Alternative PDE

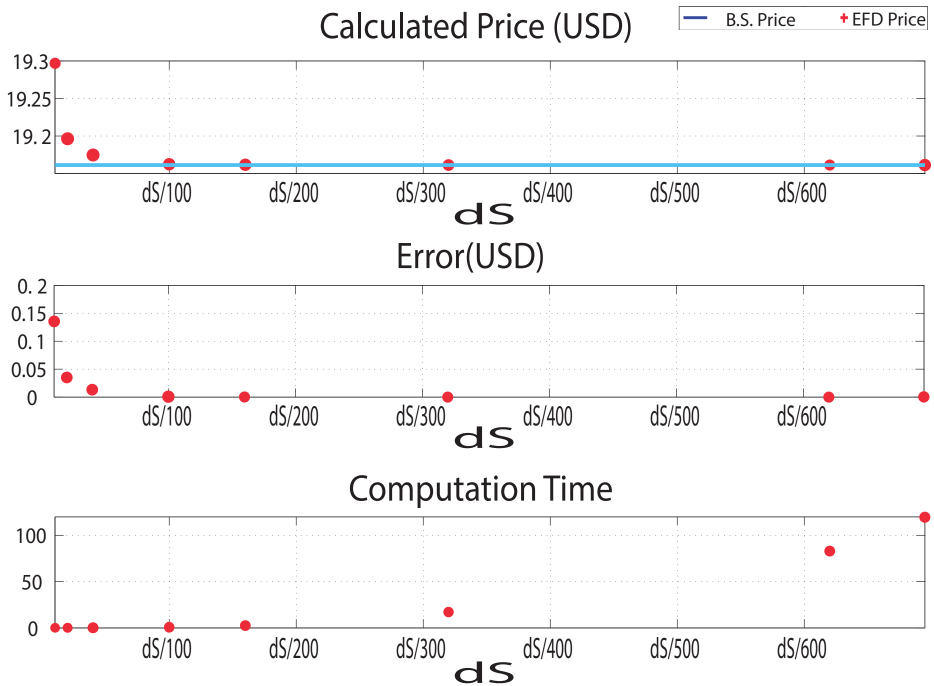

Figure 2: Numerical results for a *call* option with $r = 0.05$, $\sigma = 0.025$ and $k = 100$

Mining Projects

From [Haque et al., 2014] we've taken the PDE which describes the value for a gold mine when it's opened.

$$\frac{1}{2}P^2\sigma^2\frac{\partial^2 V}{\partial P^2} - q\frac{\partial V}{\partial Q} + (r-\delta)P\frac{\partial V}{\partial P} - (r+\lambda_c)V + q(P-C)(1-G) = 0 \quad (16)$$

dividing by $-q$ and taking $\gamma^2 = -\frac{\sigma^2}{q}$, $\rho = -\frac{r-\delta}{q}$, $\rho^* = -\frac{r-\lambda_c}{q}$ and $g = 1 - G$

Mining Projects

See that (16) can be expressed as:

$$\frac{1}{2}\gamma^2 P^2 \frac{\partial^2 V}{\partial P^2} + \frac{\partial V}{\partial Q} + \rho P \frac{\partial V}{\partial P} = \rho^* V + g(P - C) \tag{17}$$

where P is the price of gold, Q is the total reserve of gold, q is the average gold production rate, C is the total cost, G is the Corporate taxes, δ is the country risk, σ is the gold price volatility, λ_c is the convenience yield for holding gold and r is the Risk free rate.

Mining Projects

Following the methodology described before, we obtain:

$$\frac{1}{2}\gamma^2 X^2 \frac{\partial^2 H}{\partial X^2} + \frac{\partial H}{\partial Q} = (\rho^* - \rho)H + g(X - Ce^{\rho(\Phi - Q)}) \qquad (18)$$

where Φ is the maximum reserve of gold for the mine.

Schedule

Activity	Estimated Time Range
Literature Review. ✓	Jul. 21 - Aug. 8
Applying the numerical scheme to Black-Scholes' equation. Make simulations and tests. ✓	Aug. 10 - Aug. 21
Determine the convergence, consistency and stability of the numerical schema. *working on it...*	Aug. 21 - Oct. 3
Applying the numerical scheme to specifics PDE from the literature of interest. Make simulations and tests.	Oct. 3 - Oct. 15
Extra work.	Oct. 16 - Nov. 3

References I

Black, F. and Scholes, M. (1973).
The pricing of options and corporate liabilities.
The Journal of Political Economy, pages 637–654.

Haque, M. A., Topal, E., and Lilford, E. (2014).
A numerical study for a mining project using real options valuation under commodity price uncertainty.
Resources Policy, 39:115–123.

Hull, J. C. (2006).
Options, futures, and other derivatives.
Pearson Education India.

Velásquez, M., Rojas, A., and Cuscagua, M. (2015).
Diferencias finitas explícitas y otros métodos para la valoración de opciones financieras. Final report stochastics processes 2.