Study on the one-dimensional linear advection-diffusion equation with finite differences and linear finite elements methods

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***Abstract*—In this work we present the methodology and results coming from the application of the finite element method on an advection-diffusion problem modelling the air quality of a given area defined mathematically by a partial differential equation under certain domains and boundary and initial conditions. Through the consideration of certain conditions on the problem it was possible to build a discretization of the reduced and general equation using the finite difference method as a comparation base to analyze the effectivity of the main method. Finally, the concentration curves and the evolution surface representing it are presented in order to scrutinize in the obtained numerical results.**

Keywords*—*diffusion-advection problem, finite element method, finite difference method.

I. Introduction

The advection-diffusion equation we study is based on the transportation of a pollutant in the enviroment under certain conditions [4]. Such kind of problem is usually treated from the partial differential equation scheme and exact methods such as separation of variables, change of variables and transformations are usually used in order to look for exact solutions in a given domain; however, these methods cannot be applied when we deal with non-linear equations, non-homogeneous initial or boundary conditions, irregular domains or when the resources are not enough. For this reason, several numerical methods are being used in order to achieve approximate results of equations whose importance laids on the problems they represent or model. In this work we focus on two proccesses, advection and diffusion, which would allow us to build a theorical model, letting us predict, for example, how peak concentrations will change in response to prescribed changes in the domain and in the nature and source of pollution.

We think that using more accurate methods, such as the finite elements one, we can build more realistic cases applying real conditions and therefore get better results describing a problem that by today has became into a societal issue. We do not seek to add literature on a classical problem, but to contribute on the development of new theory aiming to understand and apply knowledge coming from our daily work.

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The reason why we the finite element method is applied on this kind of problem is the efficiency of the method on similar problems [6]. Nowadays its development has increased widely since its presentation at 1943 by *Richard Courant* and it is usually useful to approximate solutions of partial differential equations governing mechanical systems and it is also commonly used in the thermodicamical and engineering area. Overall, in mathematics, the finite element method is a numerical technique for finding approximate solutions to boundary value problems for partial differential equations. It uses subdivision of a whole problem domain into simpler parts, called finite elements, and variational methods from the calculus of variations to solve the problem by minimizing an associated error function. In engineering, as mentioned, this method has been one of the computational tools most used and developed over the last thirty years. Such level of development is translated in a variety of programs (softwares) of commercial distribution and the ability to simulate relevant problems in different areas of science and engineering. The relative ease of access to such commercial programs originated at the same time an excessive use of them diminishing the limitations of the method or the understanding of the problem under simulation.

On the other hand, although commercial programs are aimed to be multi purpose tools, it is not uncommon to find problems with particular conditions that clearly exhaust the software capabilities available [1]. In such cases it is necessary to modify the commercial codes by incluiding own algorithms capable of attack the problem at hand. It is clear then that the current and potential user of the finite element method requires two fundamental ingredients before tackling, evaluate or understand the engineering problem. First, a broad understanding of the problem at hand and second detailed knowledge of the capabilities and limitations of the method of solution.

Now, dealing with the advection-diffusion equation it has been only a few studies concerning on the numerical but also new theorical approach with the method, i.e, Olshanskii [8] and Bochev [2] have been able to work from the theorical aspect of this application but without going further on the stability criteria. Nevertheless it was shown that such

application is possible and has the posibility of been improved by adding computacional perfomances or changing/mixing the classical problem and its characteristics, such as domain or initial-boundary conditions. The Galerkin formulation and finite differences method have been also used to study the problem described above by Roig [10] and Friedman [4] respectively, both of which present the finite element method as a further work. Hence several authors have made important contributions to the subject over the years, specially mixing methods with the finite elements one. Ewing [3], for example, in his work *"A summary of numerical methods for time-dependent advection-dominated partial differential equations"*, shows a general prospect on how the methods and techniques have been applied on the advection-diffusion problem and also presents a light discussion on the incoming methods, all of them related with the finite elements scheme.

II. Problem Statement

In nature, advection, diffusion and other underlying processes deal with the transportation of a given chemical species in the medium [4]. Now, taking advection and diffusion only in count and letting c be the concentration of the specie as a function of position (x_1, x_2, x_3) and time t, the partial differential equation (1) describing the convection-diffusion phenomena on a physical system is found over any bounded domain D whose boundary conditions are defined on the boundary of D denoted by ∂D:

$$\begin{aligned} \frac{\partial C}{\partial t} + \nabla.(\mathbf{u}C) &= D\nabla^2 C, \\ C(\mathbf{x}, 0) &= C_0(x_1, x_2, x_3) \end{aligned} \tag{1}$$

where $\mathbf{u}$ is the chemical crossflow velocity and D the diffusion coefficient. Now, if we wanted to present Equation (1) on a more realistic way, for example, on not rectangular domains such as steady surfaces then our problem would be defined as follows.

Let Ω be an open domain in $\mathbb{R}^3$ and Γ be a connected C^2 compact surface contained in Ω. Assuming the advection velocity is everywhere tangential to the surface and having $\mathbf{w} : \Omega \to \mathbb{R}^3$ as a divergence-free velocity field in Ω then the surface advection-diffusion equation, analogous to Equation (1), takes the form:

$$u_t + \mathbf{w}.\bigtriangledown_\Gamma u - \epsilon \bigtriangleup_\Gamma u = 0 \tag{2}$$

where $\bigtriangleup_\Gamma$ denotes the *Laplace-Beltrami* operator on Γ defined as the Laplace operator extension on surfaces, i.e., the divergent of the gradient.

Equation (2) has been studied by Olshanskii [8] applying one only numerical method, the finite elements method, which under the given conditions would provide a stabilized discretization method for the surface equation, letting apply numerical experiments and get consistent results with those expected from other similar approaches like the one proposed by Bochev [2].

III. Finite Difference Method

Generally the finite difference method is considered as a numerical method for solving differential equations by approximating them with difference equations, in which finite differences approximate the derivatives. Therefore, variations on the method are taken as discretization methods and actually are the most dominant approach to numerical solutions of partial differential equations [5].

Considering the expression in (1) the way to solve it by this method is to approximate all the derivatives by finite differences. If the domain space is partionated using a mesh $\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_J$, and in time using a mesh $t_0, t_1, \cdots, t_N$, then the assumption of an uniform partition both in space and in time will generate the discrete system. The set of lattice points in the $\mathbf{x} - t$ plane is given by $\mathbf{x} = \mathbf{J} \cdot \Delta\mathbf{x}$ and $t = n\Delta n$, where $\mathbf{J} = (j_1, j_2, j_3)$ and $\Delta\mathbf{x} = (\Delta x_1, \Delta x_2, \Delta x_3)$ for $j_i = 0, \pm 1, \pm 2, \cdots$, $(i = 1, 2, 3)$, $n = 0, 1, 2, 3, \cdots$.

The approximation to $C(\mathbf{J} \cdot \Delta\mathbf{x}, n\Delta t)$ will be denoted by $C^n_{\mathbf{J}}$, noting that 'n' is not a power. The discretization is as follows:

$$\begin{aligned} \frac{\partial C}{\partial t} \quad &\text{by} \quad \frac{C(\mathbf{J}\cdot\Delta\mathbf{x}, (n+1)\Delta t) - C(\mathbf{J}\cdot\Delta\mathbf{x}, n\Delta t)}{\Delta t}, \\ \frac{\partial C}{\partial \mathbf{x}} \quad &\text{by} \quad \frac{C(\mathbf{J}\cdot\Delta\mathbf{x}, n\Delta t) - C(\mathbf{J\text{-}1}\cdot\Delta\mathbf{x}, n\Delta t)}{\Delta\mathbf{x}}, \\ \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \quad &\text{by} \quad \frac{\mathbf{u}(\mathbf{J}\cdot\Delta\mathbf{x}) - \mathbf{u}((\mathbf{J}-\mathbf{1})\cdot\Delta\mathbf{x})}{\Delta\mathbf{x}}, \quad \text{and} \\ \nabla^2 C \quad &\text{by} \\ &\frac{1}{(\Delta\mathbf{x})^2}\{C((\mathbf{J}+\mathbf{1})\Delta\mathbf{x}, n\Delta t) - 2C(\mathbf{J}\cdot\Delta\mathbf{x}, n\Delta t) \\ &\qquad + C((\mathbf{J}-\mathbf{1})\Delta\mathbf{x}, n\Delta t)\} \end{aligned}$$

or rather, $\frac{C^{n+1}_{\mathbf{J}} - C^n_{\mathbf{J}}}{\Delta t}$, $\frac{C^n_{\mathbf{J}} - C^n_{\mathbf{J\text{-}1}}}{\Delta x}$, $\frac{C^n_{\mathbf{J+1}} - 2C^n_{\mathbf{J}} + C^n_{\mathbf{J\text{-}1}}}{(\Delta x)^2}$ and $\frac{\mathbf{u_J} - \mathbf{u_{J-1}}}{\Delta\mathbf{x}}$ respectively.

Equation (1) can be expanded using indicial notation as:

$$\begin{aligned} \frac{\partial C}{\partial t} + (\hat{x_1}\frac{\partial}{\partial x_1} + \hat{x_2}\frac{\partial}{\partial x_2} + \hat{x_3}\frac{\partial}{\partial x_3}) \cdot (\mathbf{u}C) &= D\sum_{i=1}^{3}\frac{\partial^2 C}{\partial {x_i}^2} \\ \frac{\partial C}{\partial t} + \frac{\partial(\mathbf{u}C)}{\partial x_1} + \frac{\partial(\mathbf{u}C)}{\partial x_2} + \frac{\partial(\mathbf{u}C)}{\partial x_3} &= D\sum_{i=1}^{3}\frac{\partial^2 C}{\partial {x_i}^2} \\ \frac{\partial C}{\partial t} + \mathbf{u}\cdot C_{,i} + C\cdot\mathbf{u}_{i,i} &= DC_{,i} \end{aligned} \tag{3}$$

Without loss of generality we may assume $\mathbf{u}$ such that $\mathbf{u}_{i,i} = 0$, i.e.,

$$\frac{\partial u_{x_1}}{\partial x_1} + \frac{\partial u_{x_2}}{\partial x_2} + \frac{\partial u_{x_3}}{\partial x_3} = 0$$

so (3) is replaced by

$$\frac{C(\mathbf{J}\cdot\Delta\mathbf{x},(n+1)\Delta t)}{\Delta t}-\frac{C(\mathbf{J}\cdot\Delta\mathbf{x},n\Delta t)}{\Delta t}+$$
$$\mathbf{u}\cdot\frac{C(\mathbf{J}\cdot\Delta\mathbf{x},n\Delta t)}{\Delta\mathbf{x}}-\mathbf{u}\cdot\frac{C(\mathbf{J\text{-}1}\cdot\Delta\mathbf{x},n\Delta t)}{\Delta\mathbf{x}}=$$
$$D\frac{\{C((\mathbf{J}+\mathbf{1})\Delta\mathbf{x},n\Delta t)-2C(\mathbf{J}\cdot\Delta\mathbf{x},n\Delta t)\}}{(\Delta\mathbf{x})^2}+$$
$$D\frac{C((\mathbf{J}-\mathbf{1})\Delta\mathbf{x},n\Delta t)}{(\Delta\mathbf{x})^2}$$

$$\Rightarrow C_{\mathbf{J}}^{n+1}=C_{\mathbf{J}}^{n}+D\frac{\Delta t[C_{\mathbf{J+1}}^{n}-2C_{\mathbf{J}}^{n}+C_{\mathbf{J\text{-}1}}^{n}]}{\Delta\mathbf{x}^2}-\mathbf{u}\cdot\frac{\Delta t[C_{\mathbf{J}}^{n}-C_{\mathbf{J\text{-}1}}^{n}]}{\Delta\mathbf{x}} \quad (4)$$

Since a forward difference at time $t_n, n \leq N$ has been applied we get an explicit method for solving the diffusion-advection equation. Using the recurrence relation, and knowing the values at time n, it is possible to obtain the corresponding values at time $n+1$. The initial condition provee the values for $C(\mathbf{J}\cdot\Delta\mathbf{x},0)=C_{\mathbf{J}}^{0}$.

A. Numerical Stability for the Difference Scheme

In order to analyze the stability of the last procudure, i.e., check if the finite difference scheme is stable, the *von Neumann* criterion, also known as *Fourier* stability analysis, allows us to find the conditions, if possible, under which the errors made at one time step of the calculation do not cause the errors to increase as the computations are continued. For our time-dependent problem, stability would guarantee that the numerical method (4) produces a bounded solution whenever the solution of the exact differential equation (1) is bounded. If every solution to the finite difference equation having the form $C_{\mathbf{J}}^{n}=\xi^n e^{i\boldsymbol{\beta}\cdot\mathbf{J}}$, ($\boldsymbol{\beta}\in\mathbb{R}^3$, $\xi=\xi(\beta)$ complex) has the property $\mid\xi\mid\leq 1$, then the *von Neumann* criterion assures stability for the initial value problem.

The importance of determine whether keeping $\Delta\mathbf{x}$ and Δt fixed then $C_{\mathbf{J}}^{n}$ remains bounded as $n\to\infty$ uniformly in the $\mathbf{J}$'s laids on the concerning of the behavior of the solution to the discretized problem as the time variable becomes very large. However, it is also important to find out if the $C_{\mathbf{J}}^{n}$ approximate values $C(\mathbf{J}\cdot\Delta\mathbf{x},n\Delta t)$ at the mesh points in some sense as long as $\Delta\mathbf{x}$ and Δt are made sufficiently small, in such way that the finite difference scheme is convergent. Fortunately, roughly speaking, the *Lax* equivalence theorem says that if the original differential initial value problem is well-posed, and if one has done a reasonable job builing the difference scheme, then stability implies convergence.

Let us consider the case when $\mathbf{u}=(u_{x_1},u_{x_2},u_{x_3})=(U,0,0)$ so our index vectors become into $\Delta\mathbf{x}=\Delta x$, $\mathbf{J}=j$ and $\mathbf{1}=1$. The expression in (4) is then reduced to

$$\begin{aligned}C_j^{n+1}=&C_j^n-\frac{\Delta t}{\Delta x}(C_j^n-C_{j-1}^n)+\\&D\tfrac{\Delta t}{(\Delta x)^2}(C_{j+1}^n-2C_j^n+C_{j-1}^n)\end{aligned} \quad (5)$$

Applying the *von Neumann* criterion to the last difference equation by letting $C_j^n=\xi^n e^{i\beta j}$ and $r=\frac{\Delta t}{(\Delta x)^2}$ we get

$$\begin{aligned}\xi^{n+1}e^{i\beta j}=&\xi^n e^{i\beta j}-U.r\Delta x(\xi^n e^{i\beta j}-\xi^n e^{i\beta(j-1)})\\&+D.r(\xi^n e^{i\beta(j+1)}-2\xi^n e^{i\beta j}+\xi^n e^{i\beta(j-1)})\end{aligned}$$

Hence, removing the $\xi^n e^{i\beta j}$ factor,

$$\begin{aligned}\xi&=1-U.r\Delta x(1-e^{-i\beta})+D.r(e^{i\beta}-2+e^{-i\beta})\\&=1-U.r\Delta x(1-cos\beta)-iU.r\Delta x sin\beta+2D.r(cos\beta-1)\\&=\big(1-(2D.r+U.r\Delta x)(1-cos\beta)\big)-iU.r\Delta x sin\beta\end{aligned}$$

where the *Euler*'s identity has been used. Taking the magnitude of ξ we find

$$\begin{aligned}\mid\xi\mid^2&=1-2(2D.r+U.r\Delta x)(1-cos\beta)+(2D.r+U.r\Delta x)^2\\&\quad(1-cos\beta)^2+U^2r^2(\Delta x)^2sin^2\beta\\\Rightarrow\mid\xi\mid^2-1&=-2(2D.r+U.r\Delta x)(1-cos\beta)\\&\quad+4D^2r^2(1-cos\beta)^2+4D.U.r^2\Delta x(1-cos\beta)^2\\&\quad+U^2r^2(\Delta x)^2(1-2cos\beta+1)\end{aligned}$$

For $(1-cos\beta)>0$ the condition $\mid\xi\mid^2\leq 1$ is equivalent to

$-(2k+U\Delta x)+2D^2r(1-cos\beta)+2D.U.r\Delta x(1-cos\beta)+U^2r(\Delta x)^2\leq 0$

$\Rightarrow \quad r(U^2(\Delta x)^2+(2D^2+2D.U\Delta x)(1-cos\beta))\leq 2D+U\Delta x$

$\Rightarrow \quad r(U^2(\Delta x)^2+4D.U\Delta X+4D^2)=r(U\Delta x+2D)^2\leq 2D+U\Delta x$

$\Rightarrow \quad r(2D+U\Delta x)\leq 1$

These inequalities show us that if the *von Neumann* criterion holds, i.e., if $\mid\xi\mid\leq 1$ for all real β satisfying $1-cos\beta>0$ then $\mid\xi\mid^2\leq 1$.

IV. Finite Element Method

In short the finite element method results into an algorithm to solve a problem defined through differential equations and boundary conditions. Such method is built as follows:

- Formulate the problem is his variational form.
- The spacial domain must be divided by a partition into subdomains, called finite elements. A finite element vector space is constructed associated with the previous partition. The approximate numerical solution is obtained by finite elements a linear combination in that vector space.
- The projection of the original varioational problem on the finite element space coming from the partition is obtained. This gives place to a system with a finite numer of equations, but in general with a large number of unknown equations equal to the dimension of the vector space and in general, the higher the dimension better will be the numerical approximation obtained.
- The last step is the numerical computation of the solution of the equation system.

In short, the finite element method is characterized by two processes, the choosing of a grid for the domain and of the basis functions.

In practical terms, the finite element method (FEM) is method for numerical solution of field problems. Cutting

through a structure into several elements or pieces of the structure then reconects them at specific "nodes", resulting into a set of simultaneous algebraic equations $[\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{F}\}$, called property, behavior and action respectively, whose nature depends on the problem one. However, in order to make the algebraic equations easier for the entire domain several techniques are used, such as the division of the domain into a number of small simple elements, the interpolation of a field quantity by a polynomial over an element or the imposition of adjacent elements sharing the degrees of freedom at connecting nodes [1].

Posing the unsteady linear advection-diffusion equation in (1) under the conditions in Section (III-A) and limiting our spacial and temporal domains as $\mid x \mid < 1$ in x and $t \in [0, T]$ in t, we get the following relation:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2}, \quad -1 < x < 1, \quad t \in [0, T], \qquad (6)$$

where C, u and D are as described before. Imposing *Dirichlet* boundary conditions $C(-1, t) = C(1, t) = 0$ and the initial condition $C(x, 0) = sin\pi x$ will let us apply the method and compare with known solutions for the same problem [7].

Assume C is a solution of the differential equation (6). Multiplying by an arbitrary test function v and integrating on $[-1, 1]$ we get:

$$\int_{-1}^{1} \left((\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x})v - D.v\frac{\partial^2 C}{\partial x^2} \right) dx = 0$$

Integrating by parts (i.e., using *Green*'s formula) the last expression is transformed into:

$$\int_{-1}^{1} \left((\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x})v + D\frac{\partial C}{\partial x}\frac{\partial v}{\partial x} \right) dx = 0 \qquad (7)$$

The global finite elements approximation is sought in terms of linear *Lagrange* polynomials making up hat functions such that the numerical discretization is

$$C_h(x) = \sum_{i=1}^{\epsilon+1} C_i(t)\varphi_i(x), \quad -1 \le x \le 1, \qquad (8)$$

where h refers to the finite element grid size. i denotes the index of the finite element grid nodes and ϵ the number of elements covering the spacial domain and therefore $N = \epsilon + 1$ is the number of grid points. C_i are the time dependent nodal values. The hat functions are linear interpolants with the following definition.

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h} & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{h} & x \in [x_i, x_{i+1}], \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases} \quad 2 \le i \le \epsilon.$$

Replacing the approximate formulas in (7), we obtain:

$$\int_{-1}^{1} \left((\frac{\partial C_h}{\partial t} + u\frac{\partial C_h}{\partial x})v_h + D\frac{\partial C_h}{\partial x}\frac{\partial v_h}{\partial x} \right) dx = 0 \qquad (9)$$

Choosing the test functions as the approximation polynomials, i.e., $v_h = \varphi_i$ for $i = 1, \ldots, \epsilon$, and using (9) we get:

$$\int_{-1}^{1} \left((\frac{\partial C_h}{\partial t} + u\frac{\partial C_h}{\partial x})\varphi_k + D\frac{\partial C_h}{\partial x}\frac{\partial \varphi_k}{\partial x} \right) dx = 0$$

where the subscript k corresponds to the sumation coming from (8) so that

$$\int_{-1}^{1} \left((\sum_{k=2}^{N-1} \varphi_i \tfrac{dC_k}{dt} + u\tfrac{d\varphi_i}{dx}C_k)v_k + D\sum_{k=2}^{N-1} \tfrac{d\varphi_i}{dx}C_k\tfrac{d\varphi_k}{dx} \right) dx = 0$$

$$\sum_{k=2}^{N-1} \int_{-1}^{1} \left((\varphi_i \tfrac{dC_k}{dt} + u\tfrac{d\varphi_i}{dx}C_k)\varphi_k + D\tfrac{d\varphi_i}{dx}C_k\tfrac{d\varphi_k}{dx} \right) dx = 0$$

$$\sum_{k=2}^{N-1} \int_{-1}^{1} \left(\varphi_i\varphi_k\tfrac{dC_k}{dt} + u\tfrac{d\varphi_i}{dx}\varphi_k C_k + D\tfrac{d\varphi_i}{dx}\tfrac{d\varphi_k}{dx}C_k \right) dx = 0$$

$$\textstyle\sum_{k=2}^{N-1} \left(\int_{-1}^{1} \varphi_i\varphi_k\tfrac{dC_k}{dt}dx + u\int_{-1}^{1} \tfrac{d\varphi_i}{dx}\varphi_k C_k dx + D\int_{-1}^{1} \tfrac{d\varphi_i}{dx}\tfrac{d\varphi_k}{dx}C_k dx \right) = 0$$

$$\textstyle\sum_{k=2}^{N-1} \left(\int_{-1}^{1} \varphi_i\varphi_k\tfrac{dC_k}{dt}dx + (u\int_{-1}^{1} \tfrac{d\varphi_i}{dx}\varphi_k dx + D\int_{-1}^{1} \tfrac{d\varphi_i}{dx}\tfrac{d\varphi_k}{dx}dx)C_k \right) = 0$$

$$\sum_{k=2}^{N-1} \left(M_{i,k}\tfrac{dC_k}{dt} + \big(uR_{i,k} + DK_{i,k}\big)C_k \right) = 0$$

where the mass matrix $[M]$, weak derivative matrix $[R]$ and stiffness matrix $[K]$ corresponds respectively to

$$M_{i,k} = \int_{-1}^{1} \varphi_i\varphi_k dx, \quad 2 \le i \le N-1,$$
$$R_{i,k} = \int_{-1}^{1} \tfrac{d\varphi_i}{dx}\varphi_k dx, \quad 2 \le i \le N-1,$$
$$K_{i,k} = \int_{-1}^{1} \tfrac{d\varphi_i}{dx}\tfrac{d\varphi_k}{dx}dx, \quad 2 \le i, k \le N-1.$$

Note that a linear system of algebraic equations of order $N-2$ has been generated as the end points are given boundary values, i.e., $k = 1$ and $k = N$ correspond to $C(-1, t)$ and to $C(1, t)$ correspondingly.

For a given node i its discrete equation is then given by

$$\begin{gathered} \tfrac{h}{6}\tfrac{dC_{i-1}}{dt} + \tfrac{2h}{3}\tfrac{dC_i}{dt} + \tfrac{h}{6}\tfrac{dC_{i+1}}{dt} \\ +u\tfrac{C_{i+1} - C_{i-1}}{2} - \tfrac{D}{h}(C_{i-1} - 2C_i + C_{i+1}) = 0. \end{gathered}$$

Using an implicit *Crank-Nicolson* time scheme for the viscous part corresponding to those terms related with $[K]$ and a second-order *Adams-Bashforth* explicit integration for the advection term on the set of ordinary differential equations $[M]\frac{d\underline{C}(t)}{dt} + ([K] + [R])\underline{C}(t) = 0$, where $\underline{C}$ is the vector collecting all the problem unknowns, then the full discrete equations read with Δt depending on the *Courant-Friedrich-Levy* condition for the explicit part as follows:

$$\Big([M] + \tfrac{\Delta t}{2}[K]\Big)\underline{C}^{n+1} = \Big([M] - \tfrac{\Delta t}{2}[K]\Big)\underline{C}^{n} + \tfrac{\Delta t}{2}[R]\big(3\underline{C}^{n} - \underline{C}^{n-1}\big) \qquad (10)$$

V. Results

A. Diffusion-Advection Equation for P2 Problem

Let us consider the case when $\mathbf{u} = (U, V, 0) = (-\sin\theta, \cos\theta, 0)$, therefore $\Delta\mathbf{x} = (\Delta x_1, \Delta x_2) = (\Delta x, \Delta y)$ and $\mathbf{J} = (j_1, j_2) = (j, l)$. For the initial condition and conditions given below we apply the differential scheme (4) for $\mid x \mid \leq 100$ and $\mid y \mid \leq 100$ and different points of time.

$$c_0(x,y) = \begin{cases} 50(1+\cos\frac{\pi R}{4}) & \text{, if } R < 4 \\ 0 & \text{, if } R > 4 \end{cases}$$

$$\text{where} \begin{cases} \theta & = \arctan\frac{y}{x} \\ R^2 & = (x-x_0)^2 + (y-y_0)^2 \\ (x_0, y_0) & = (5, -10) \end{cases}$$

Note that $\mathbf{u}$ is such that satisfy the condition $\nabla \cdot \mathbf{u} = 0$, indeed, since

$$(U,V) = \left(-\sin\theta, \cos\theta\right) = \left(-\frac{y}{\sqrt{(x^2+y^2)}}, \frac{x}{\sqrt{x^2+y^2}}\right),$$

thus we obtain

$$\begin{aligned} \nabla \cdot \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x} + \frac{\partial \mathbf{u}}{\partial y} &= -y\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x^2+y^2}}\right] + x\frac{\partial}{\partial y}\left[\frac{1}{\sqrt{x^2+y^2}}\right] \\ &= \frac{yx}{(x^2+y^2)^{\frac{3}{2}}} - \frac{yx}{(x^2+y^2)^{\frac{3}{2}}} \\ &= 0. \end{aligned}$$

Letting $k = 1$, $\Delta x = \Delta y = 1$, $\Delta t = 2.5 * 10^{-1}$ and 100 nodes for each axis we obtain the concentration surfaces for $t = 1.25s, 6.25s, 15s, 30s$ respectively using the routines in [9].

B. Diffusion-Advection Equation for P1 Problem

For purposes of testing the finite element discretization presented in the last section we stablish the equation parameters D and u as 1 and $\frac{1}{10\pi}$ respectively under the boundary and initial conditions as mentioned before. Corresponding to the method Δt is set as $\frac{1}{100}$ with $\epsilon = 1000$ elements. Applying the scheme (10) we obtain the curves in Figs. (V-B) for equally spaced domain with $\Delta x = 0.2$.

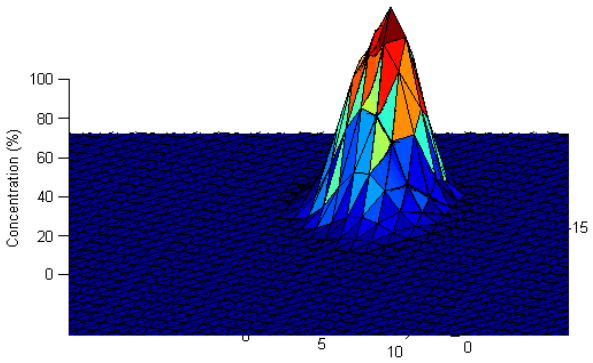

Fig. 1. Concentration surface evolution for 1.25 secs.

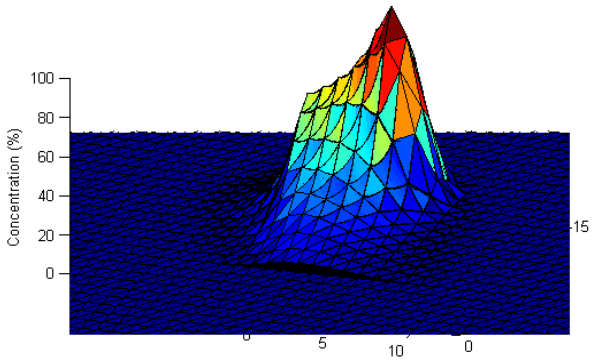

Fig. 2. Concentration surface evolution for 6.25 secs.

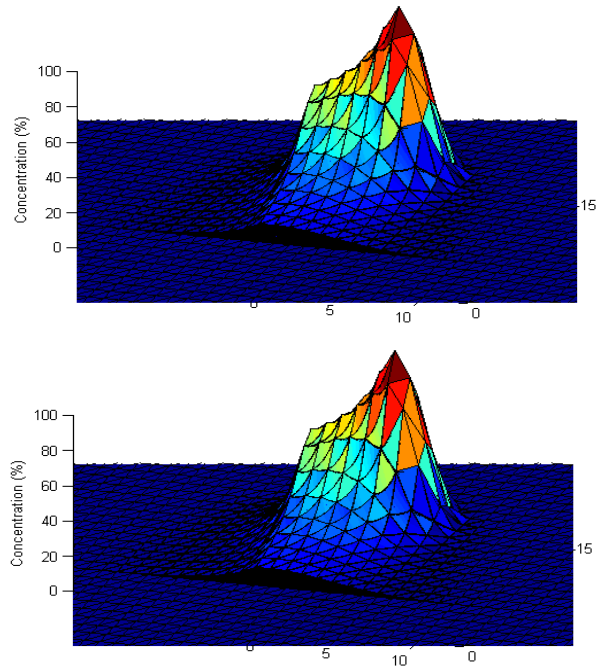

Fig. 3. Concentration surface evolution for 15 and 30 secs.

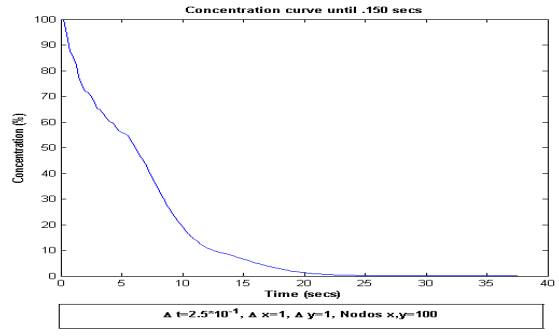

Fig. 4. Concentration percentage evolution curve for different parameters.

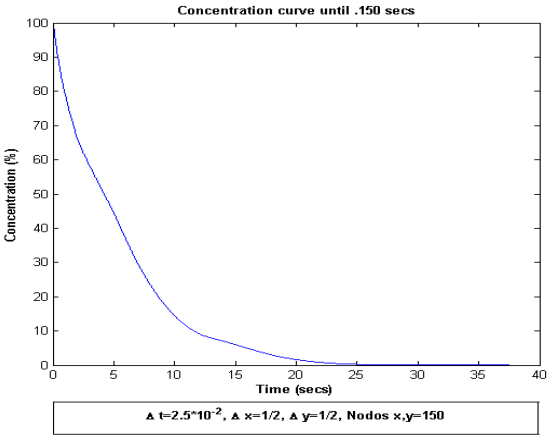

Fig. 5. Concentration percentage evolution curve for different parameters.

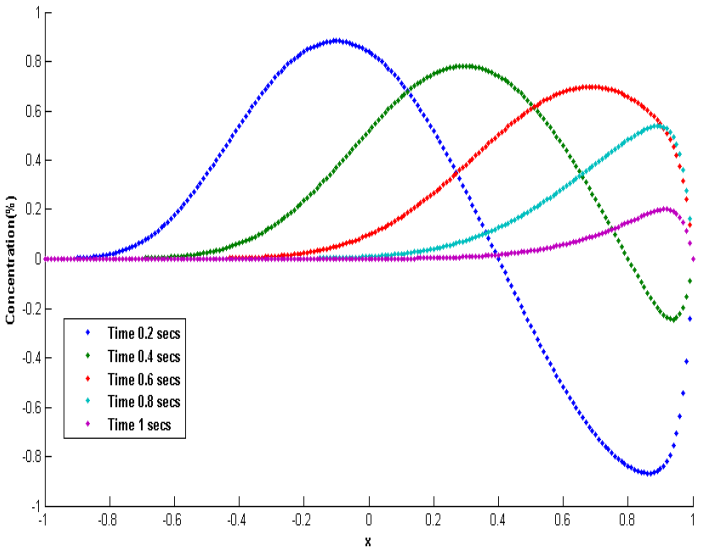

Fig. 6. Concentration percentage curves for different times with finite element method.

VI. Concluding Remarks

The finite elements method was applied on the one-dimensional linear advection-diffusion equation for Dirichlet boundary conditions and a sinusoidal initial condition after had realised a theorical analysis using the finite difference method, allowing us to obtain a preliminary insight into the behavior of the solution for (6). Although the presence of different parameters in the building of the solution might have a significant effect on its stability, the convergence of both methods under different sets of parameters as exposed in Figure(V-B) and Figure(V-B), guarantees a dependence only on the spatial and temporal discretization. The *von Neumann* stability criterion provees a tool to ensure degrees of stability; however, this same criterion cannot be applied for the finite element method since only when under certain conditions concerning completeness and the good behavior of the approximate solution are satisfied then convergence is insured. The accomplished results could be used to analyze the polutant concentration distribution for a certain domain, as long as the real conditions can be adequate to the theorical ones.

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