Real Options Valuation for Mining Projects Using the Finite Difference Method

J. Mauricio Cuscagua-Lopez Fredy H. Marín-Sanchez

Research Practise 1: Final Presentation
Mathematical Engineering

November 2015

UNIVERSIDAD EAFIT®

Some Definitions

- **Option:** contract which gives the buyer - the owner or holder - the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price on or before a specified date.
- **Real Option:** contract which gives the buyer the right — but not the obligation — to undertake certain business initiatives

Why to use Options?

- **Because of its versatility:** They allow for positive movements although the market does not tend to rise.
- **To ensure investment:** Minimize the risk and losses (not eliminate).

Option Valuation

Black, Scholes and Merton assumed that the price of the underlying can be modeled as a Geometric Brownian motion, then, the price S_t satisfies the stochastic differential equation (SDE) bellow:

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{1}$$

where B_t is a unidimensional standard Brownian motion (USBM).

Option Valuation

- The price of the underlying is a Geometric Brownian motion.
- No transaction costs.
- The assets are perfectly divisible.
- The underlying pays no dividends during the life of the option.
- No arbitrage opportunities.
- The negotiation of assets is continuing.
- Free interest rate risk r is constant for all maturities.

Option Valuation

Let f be the price of a call option of European type. Using the assumptions described before and the Ito's lemma [Mao, 2007] it is shown that $f(t, S)$ must satisfies the following partial differential equation (PDE):

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} = rf \tag{2}$$

where f is the price of the Option, S is the price of the underlying, σ is the volatility of the underlying related to the commodity and r is the free interest rate risk.

Boundary Conditions

For a call option of European type, the boundary conditions are:

- For $t = \tau$, then $f = max(S - K, 0)$
- For $S = S_{max}$, then $f = max(S_{max} - K, 0)$
- For $S = 0$, then $f = 0$

Where $T = \tau - t$ is the maturity time and K is the exercise price of the option.

PDE for Mining Projects Valuation

Considering the arguments reported in [Haque et al., 2014], Haque, Aminul and Topal assumed that the price of mining project can be modeled as a USBM, then, the price P_t satisfies the stochastic differential equation:

$$\frac{dP}{P} = (r - \delta)dt + \sigma d_W \tag{3}$$

where P is the spot unit price of the underlying, r is the risk free rate of interest, δ is the mean convenience yield on holding one unit of gold, σ is the volatility of returns of P and d_W is the wiener increment Standard.

PDE for Mining Projects Valuation

This project leads to a cash flow $q(P-C)(1-G)dt-\delta(\partial V/\partial P)Pdt$, where C is the total cost per unit of gold and G is the total tax. Therefore, the total return on the portfolio is:

$$dV-\frac{\partial V}{\partial P}dP+a(P-C)(1-G)dt-\delta\frac{\partial V}{\partial P}Pdt \tag{4}$$

PDE for Mining Projects Valuation

Applying the Ito's Lemma we obtain:

$$\frac{1}{2}P^2\sigma^2\frac{\partial^2 V}{\partial P^2} - q\frac{\partial V}{\partial Q} + (r-\delta)P\frac{\partial V}{\partial P} - (r+\lambda_c)V + q(P-C)(1-G) = 0 \quad (5)$$

where P is the price of gold, Q is the total reserve of gold, q is the average gold production rate, C is the total cost, G is the Corporate taxes, δ is the convenience yield for holding gold, λ_c is the country risk, σ is the gold price volatility and r is the Risk free rate.

Boundary Conditions

The boundary conditions for this PDE are:

- For $Q = 0$, then $V = 0$, i.e. for a reserve of 0, the value of the mine is 0.
- For $P = 0$, then $V = 0$, i.e. for a gold price equal to 0, the value of the mine is 0.
- For $P = P_{max}$, then $V = P_{max}Q$, i.e. for a maximum gold price, the value of the mine is that price times the reserve in the mine.

Dynamic

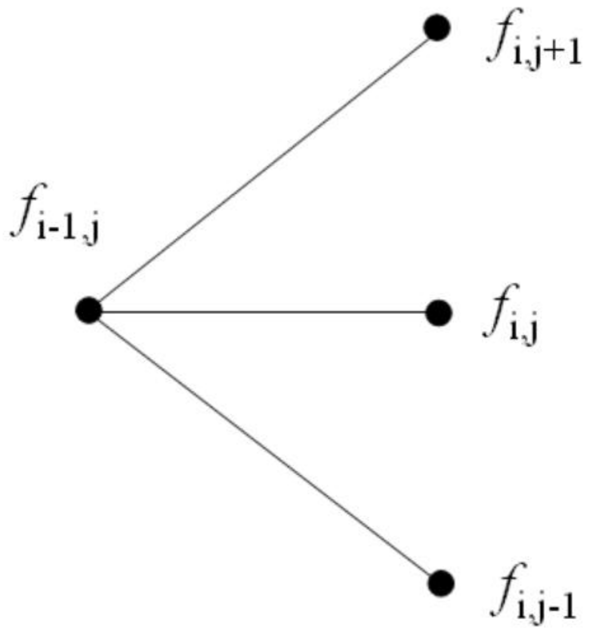

Figure 1: Dynamic of the Finite Difference method

Approximations

The Finite Difference method sets the following expressions to sample the derivatives [Hull, 2006].

$$
\begin{aligned}
\frac{\partial f}{\partial t} &\approx \frac{f_{i+1,j} - f_{i,j}}{\Delta t} \\
\frac{\partial f}{\partial S} &\approx \frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta S} \\
\frac{\partial^2 f}{\partial S^2} &\approx \frac{f_{i+1,j+1} + f_{i+1,j-1} - 2f_{i+1,j}}{\Delta S^2}
\end{aligned}
\tag{6}
$$

A Black-Scholes Model Variation

(5) has the same structure than the Black-Scholes model when we set $G = 1, \delta = \lambda_c = 0$ and $q = -1$. This is mathematically correct but without any sense (financially talking). Assuming the value of the parameters described above, we get in (5):

$$\frac{1}{2}P^2\sigma^2\frac{\partial^2 V}{\partial P^2} + \frac{\partial V}{\partial Q} + rP\frac{\partial V}{\partial P} = rV \tag{7}$$

Variable Changes

Let us take the following change of variable

$$
\begin{aligned}
H &= \mathrm{e}^{r(\tau - t)} f \\
X &= \mathrm{e}^{r(\tau - t)} S
\end{aligned}
\tag{8}
$$

where $H : H(t, X)$ and $\tau - t$ is the maturity time.

Derivatives Equivalences

By deriving and applying the chain rule to the expressions in (8) we obtain:

$$\frac{\partial f}{\partial t} = e^{-r(\tau - t)}(-rX\frac{\partial H}{\partial X} + \frac{\partial H}{\partial t}) + rHe^{-r(\tau - t)} \tag{9}$$

$$\frac{\partial f}{\partial s} = \frac{\partial H}{\partial X} \tag{10}$$

$$\frac{\partial^2 f}{\partial s^2} = e^{r(\tau - t)}\frac{\partial^2 H}{\partial s^2} \tag{11}$$

Transformed Black-Scholes Model

Using the results in (9), (10) and (11) in (2) and organizing the terms we have:

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 H}{\partial X^2} = 0 \tag{12}$$

The equivalent boundary conditions for this alternative PDE with the original one are:

- For $t = \tau$, then $f = \mathrm{e}^{r(\tau - t)} max(S - K, 0)$.
- For $S = S_{max}$, then $f = \mathrm{e}^{r(\tau - t)} max(S_{max} - K, 0)$.
- For $S = 0$, then $f = 0$.

Transformed PDE for a Mining Project

Using the equivalences between the partial derivatives, we take S and t as P and Q respectively. The alternative PDE which corresponds to the transformation of (5):

$$\frac{1}{2}\gamma X^2 \frac{\partial^2 H}{\partial X^2} + \frac{\partial H}{\partial Q} = (\rho_2 - \rho_1)H + g(X - Ce^{\rho(\Phi - Q)}) \tag{13}$$

where Φ is the maximum reserve of gold for the mine, $\gamma = -\frac{\sigma^2}{q}$, $\rho_1 = -\frac{r-\delta}{q}$, $\rho_2 = -\frac{r-\lambda_c}{q}$ and $g = 1 - G$.

Parameters for Simulation

Parameter	**Value**	**Parameter**	**Value**
σ	0.22271	G	0.3
r	0.06	q	89155
λ_C	0.03	Q	285620
δ	0.03	C	141.71

Table 1: Set of parameters for the simulation. Taken from [Haque et al., 2014]

Numerical Scheme

Using the approximations in (9 - 11), we obtain the numerical schema corresponding to (13):

$$H_{i,j} = a_j H_{i+1,j+1} + b_j H_{i+1,j} + c_j H_{i+1,j-1} + d_j \qquad (14)$$

where, $Q_i = i\Delta Q - Q$ for $i = 1, 2, ..., I$, $P_j = j\Delta P$ for $j = 1, 2, ..., J$.

Coheficients

The weights for (14) are:

$$\begin{aligned} a_j &= \frac{1}{2}\frac{\Delta Q \gamma j^2}{1+\rho_2-\rho_1} \\ b_j &= \frac{1}{2}\frac{1-\Delta Q \gamma j^2}{1+\rho_2-\rho_1} \\ c_j &= \frac{1}{2}\frac{\Delta Q \gamma j^2}{1+\rho_2-\rho_1} \\ d_j &= \frac{\Delta Q (j\Delta X - Ce^{\rho_1(\Phi-Q)})}{1+\rho_2-\rho_1} \end{aligned} \tag{15}$$

Relation Between ΔQ and ΔX

As we seek for the positivity of the weights, b_j gives us the information about the relation between the deltas.

$$\Delta Q = \frac{Q}{\left\lceil \frac{\Delta^2}{X_{max}^2 \gamma} \right\rceil} \tag{16}$$

Convergence

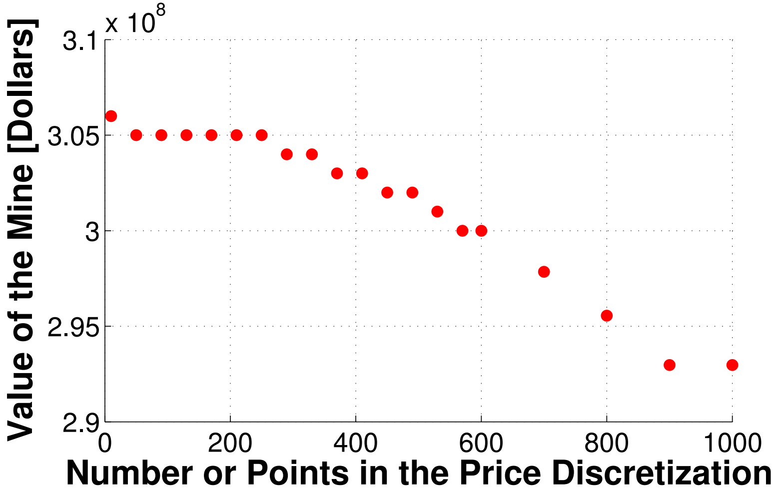

Figure 2: Convergence curve for increasing points of discretization

Computation Time

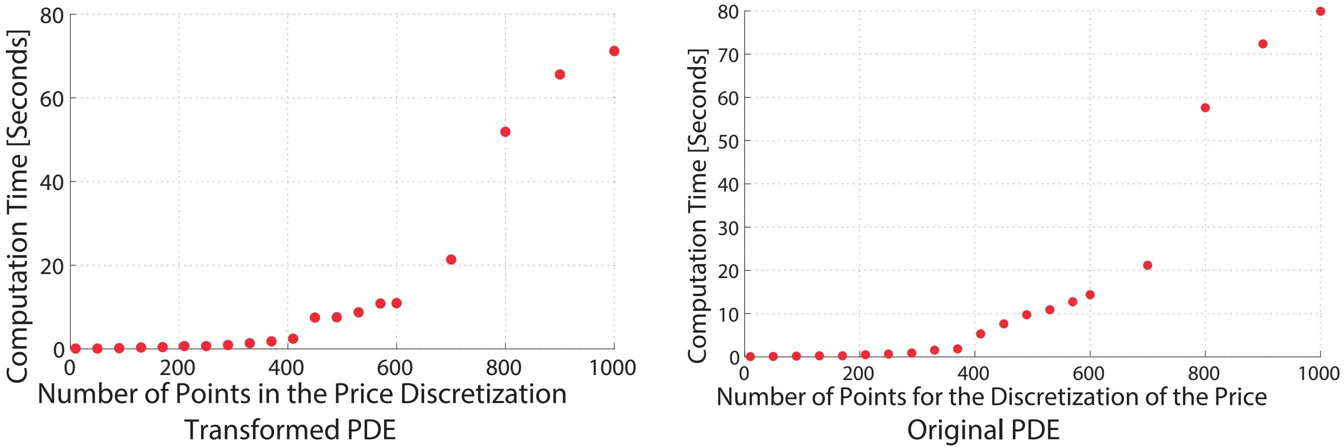

Figure 3: Comparing computation time between the transformed and original PDE for increasing points of discretization

Analysis of the Risk Free Interest Rate r

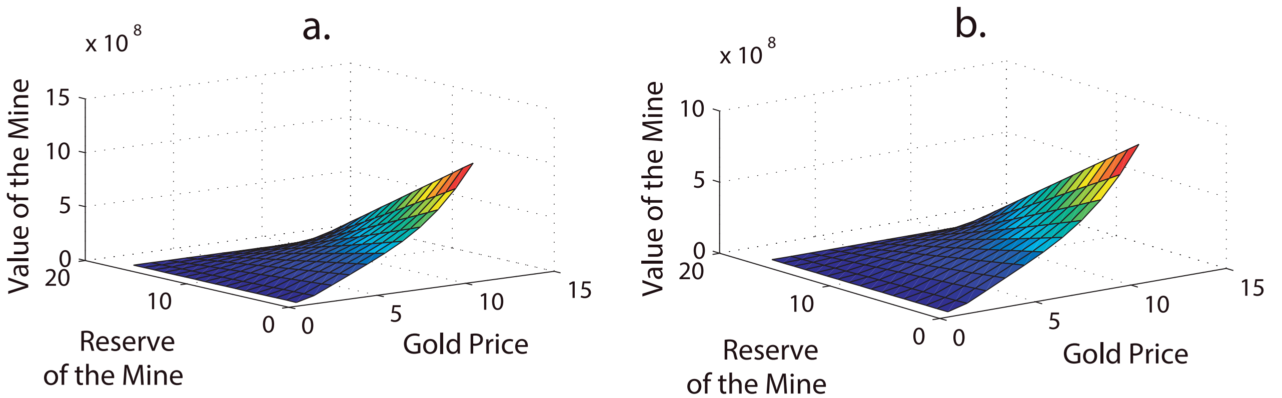

Figure 4: Mail of the mine value for different values of the free risk interest rate, *a.* $r = 0.1$, *b.* $r = 0.06$.

Analysis of the Risk Free Interest Rate r

r	**Maximum value of the mining project**
0.1	$ 1,072,265,450.9341
0.06	$ 943,299,247.8578
0.01	$ 803,680,548.3382

Table 2: Maximum value of the mining project for different values of the free risk interest rate.

Analysis of the Volatility σ

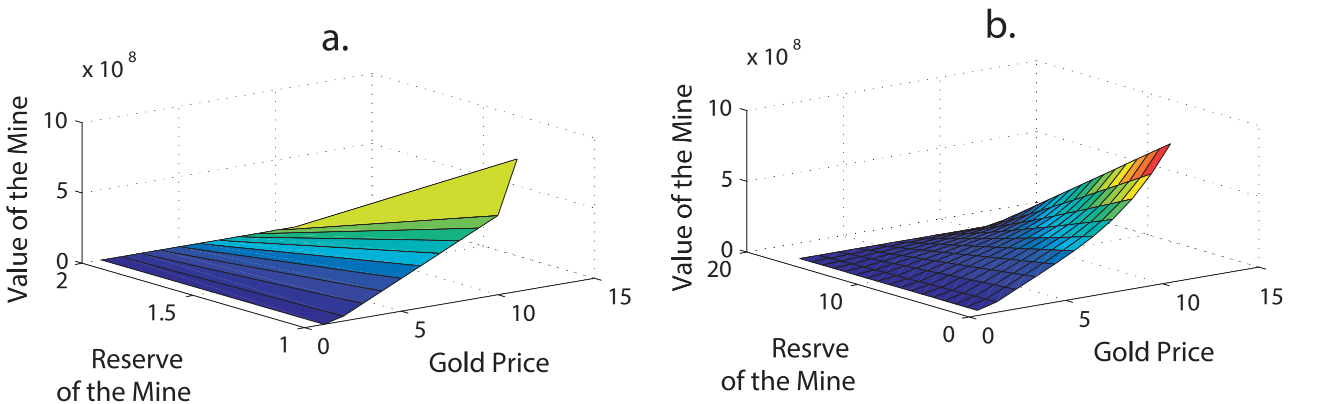

Figure 5: Mail of the mine value for different values of the volatility of the underlying, *a.* $\sigma = 0.01$, *b.* $\sigma = 0.22271$.

Analysis of the Volatility σ

σ	**Maximum value of the mining project**
0.01	\$ 563,087,328.7043
0.22271	\$ 752,231,420.0460
0.5	\$ 807,581,453.0986

Table 3: Maximum value of the mining project for different values for the volatility of the underlying.

Conclusions

- The transformation can be implemented with derivatives models of Black-Scholes model that are used for calculating several financial assets.
- It was tested the natural direct relationship between a financial asset and the volatility of the underlying and the free risk interest rate.
- The transformation makes the computation time of the calculation faster. That is of interest to a brokerage firm.

Further Work

- Test the proposed numerical scheme in several different variations of the Black-Scholes model for the calculation of different kind of financial assets.
- Verify the monotonicity, positivity, consistency, stability and convergence of the numerical scheme.
- Implement historical data to compare specific results obtained by different calculation methods and the results given by our proposed numerical scheme.

References I

Haque, M. A., Topal, E., and Lilford, E. (2014).
A numerical study for a mining project using real options valuation under commodity price uncertainty.
Resources Policy, 39:115–123.

Hull, J. C. (2006).
Options, futures, and other derivatives.
Pearson Education India.

Mao, X. (2007).
Stochastic differential equations and applications.
Elsevier.