Affine term structure models: forecasting the Colombian yield curve

Research practice II: Project proposal

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I. Problem formulation

The term structure of interest rates (TS for short) shows the relationship between interest rates and investment time horizons at a given time. Approximating and forecasting the TS is useful for pricing financial instruments and managing risk.

The most common way of representing the TS is using the yield curve. A yield curve is obtained by plotting the yields of different bonds with similar credit risk (e.g. issued by the same institution) against their maturities.

Affine term structure models (ATSMs) approximate the yield curve as a function of N factors which can be observable (macroeconomic variables) or latent. We will denote the yield of a zero-coupon bond with maturity τ by γ_τ. ATSMs assume [1] that the short rate is an affine function [1] of the state factors vector $X(t)$:

$$r = lim_{\tau\to 0}\gamma_\tau = \delta_0 + \delta_1^\top X(t) \tag{1}$$

With $\delta_0 \in \mathbb{R}$, $\delta_1 \in \mathbb{R}^N$.

The state vector is assumed to follow an affine diffusion process under the risk-neutral probability measure Q:

$$\begin{aligned} dX(t) &= \mu^Q(X)dt + \sigma(X)dW^Q(t) \\ &= (\widetilde{A} - \widetilde{B}X(t))dt + \Sigma\sqrt{S(t)}dW^Q(t) \end{aligned} \tag{2}$$

Where $\widetilde{B}$, $\Sigma \in \mathbb{R}^{N\times N}$, $\widetilde{A} \in \mathbb{R}^{N\times N}$, W is an N-dimentional independent brownian motion and $S(t)$ is a $N\times N$ diagonal matrix with entries:

$$[S(t)]_{i,i} = \alpha_i + \beta_i^\top X(t) \tag{3}$$

$\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}^N$.

The market price of risk $\Lambda(X) \in \mathbb{R}^N$ must also be specified in order to obtain the physical measure (P) dynamics:

$$\begin{aligned} dX(t) &= \mu^P(X)dt + \sigma(X)dW^P \\ &= \{\mu^Q(X) + \sigma(X)\Lambda(X)\}dt + \sigma(X)dW^P \end{aligned} \tag{4}$$

Under this structure, it can be shown [2] that the yield for any maturity τ can be obtained as an affine function of the state vector:

$$\gamma_\tau(t) = A(\tau) + B(\tau)^\top X(t) \tag{5}$$

ATSMs have been widely used in various countries over the last decade. However, this hasn't been the case for Colombia. The variety of studies that have used ATSMs for forecasting yields, estimating unobservable information (derivative prices, inflation risk premiums, etc) and grasping the TS lead us to believe that understanding and implementing this class of models for the Colombian setting could be of great use.

Therefore, we want to implement and estimate some of the most used ATSMs (with different numbers of factors

[1] A function $F : R^N \to R^M$ is said to be affine if $F(X) = A + B * X$ for some vector A and matrix B

and specifications of μ^Q, σ and Λ) using Colombian yield data. We will then assess and compare the capacity of the different models to forecast the Colombian yield curve.

The knowledge acquired through the Mathematical Engineering program will be needed for this task. We identify the following aspects of the student's formation that will be useful for the project:

- Stochastic processes, probability theory and real analysis are needed in order to fully understand ATSMs.
- Statistics, optimization, heuristics and experimental modeling need to be applied in the calibration of the models.
- Knowledge on financial instruments and markets is needed in order to understand the context of the models.

II. Objectives

A. General objective

The general objective of the project is to evaluate the performance of various ATSMs in forecasting the Colombian term structure of interest rates.

B. Specific objectives

The following objectives must be met in order to achieve our general goal:

- Understanding the theory behind ATSMs as well as their computational implementations.
- Being able to replicate empirical applications of ATSMs for the Colombian market.
- Implementing various commonly used models with Colombian yield data.
- Testing the implemented models' accuracy in forecasting the Colombian term structure.

III. Preceding research

In 1996, Duffie and Kan [2] study a class of dynamic term structure models in which a set of yields with fixed maturities are taken as the factors. They show how every yield can be obtained as an affine function of the factors, and generalize their results to latent factor models (thus naming the models "affine factor models" which would later come to be known as ATSMs). Some advantages of ATSMs are discussed: consistency with the no-arbitrage hypothesis, the possibility of pricing derivative instruments and their computational tractability.

Dai and Singleton [3] later classify ATSMs according to the number of state variables that determine the conditional volatility of the term structure. They also propose a canonical representation for ATSMs, outlining admissibility restrictions for parameters. Finally they compare and discuss the fit of different models using US dollar swap data.

Estimating ATSMs has posed a challange due to their likelihood functions only being known in closed form for a few special cases. This has led to the application of numerous methods including: *quasi maximum likelihood*, Kalman filtering, simulation, method of moments and numerical methods. We now present a brief review of some relevant studies.

Duan and Simonato [4] propose an state-space representation of ATSMs. They approximate the conditional mean and variance under the assumption that the diffusion process is gaussian and estimate the latent factors using the Kalman filter. This allows them to evaluate the likelihood function (*quasi likelihood* function for non gaussian models) and estimate the parameters of various test models.

Brandt and He [5] argue that the *quasi maximum likelihood* method is skewed for multi-factor models. They then present a correction for the *quasi-likelihood* function, which is obtained by simulation and converges to the real likelihood function. This methodology reduces the skewness and variability of the estimated parameters, but it is computationally intensive.

Ait-Sahalia and Kimmel [6] estimate various ATSMs using closed-form expansions of their likelihood functions, which are obtained using a methodology previously presented by Ait-Sahalia [7]. The obtained estimates surpass other methods' precision and allow for statistical testing. This technique has been widely used in recent years for its precision and efficiency.

ATSMs have been used for a great variety of purposes because of their flexibility and consistency with the no-arbitrage hypothesis. The fact that they model the dynamics that drive yield changes has also allowed researches to make forecasts, interpretations and hypotheses about the behavior of the TS. Some applications of ATSMs include the ones presented in the following paragraphs.

Pricing derivative financial instruments, for example, has been possible using ATSMs. Singleton and Umanstev [8] price options and swaptions on coupon bonds. Ho, Huang and Yildirim [9] also propose a methodology for pricing inflation-indexed derivatives.

Decomposition of yields into their unobservable components has also been a notable application. For instance, Duffie [10] expresses yields as a sum of the short rate and a term premium in order to analyze the behavior of expected excess returns. Durham [11] uses ATSMs to model the nominal U.S treasury and TIPS yield curves in order to estimate inflation risk premiums.

In the Colombian setting, most approaches to the TS have been oriented to interpolation and curve-fitting of cross-sectional data. Nelson-Siegel and cubic splines have been the most used methodologies for approximating unobserved yields. Only a few recent studies have tried to adjust and calibrate dynamic models.

Restrepo-Tobón and Botero-Ramírez [12] find that two classes of one-factor models can represent the Colombian TS closely. They utilize both models to price hypothetical derivative instruments but point out difficulties in estimating volatility parameters. This due to the lack of development and the unavailability of information in the Colombian market. They leave the problem of implementing, estimating and testing multi-factor models open.

Castaño, Rueda and Robayo [13] use a reparametrization of the Nelson-Siegel curve and apply the Kalman filter to estimate its parameters. Parameter estimates differ considerably from those reported by *Infoval* (Colombian price, rates and margins provider); however, the zero-coupon yield curves obtained with both sets of parameters are very close. The study also reports accurate one-period forecasts using this approach.

IV. Justification

Modeling and understanding the TS has different purposes. It allows to approximate and forecast different interest rates to value financial instruments. The TS also contains information about the current and future states of financial markets and the economy. Developments in this area can be beneficial for investors, portfolio managers and policymakers.

Interpolation and curve-fitting methods approximate unobserved yields for a given day by adjusting smooth curves to observable data obtained in the same or previous days. While this technique provides traders and managers with useful tools for day-to-day pricing, it falls short when trying to forecast or explain the behavior of the TS. Dynamic models (as are ATSMs) can fulfill this purposes and could aid investment decisions and planning.

Dynamic modeling of the TS can also enrich the knowledge about the factors that drive yield changes. The state variables can often times be interpreted and related to macroeconomic variables (see Cortés and Ramos [14] and Ochoa [15]). The models can explicitly include these macroeconomic variables. This higher understanding on the TS could aid governmental entities in processes as bond emissions and interest rate regulations.

Another benefit of ATSMs is that they are consistent with the no-arbitrage hypothesis [2]. This is guaranteed by the method for obtaining $A(\tau)$ and $B(\tau)$ from equation 5. No-arbitrage is essential for derivative pricing and also a desirable property for TS models, which commonly used methods (Nelson-Siegel and splines) do not have.

Finally, ATSMs are a useful for pricing derivative instruments, as was discussed in section III. In 2006, Arboleda [16] analyses the perspectives of the derivative market in Colombia. The study concludes consolidating a derivative market is a difficult task mostly due to the following obstacles:

- Monopolization of information by a few market participants.
- Lack of technological infrastructure.
- Lack of knowledge about the theoretical and practical aspects of financial derivatives.

We think this project could aid in diminishing these obstacles, as it intends to implement and present ATSMs in a clear way, using Colombian data.

This project seeks to be a step towards the development of the Colombian financial market. It aims to understand and implement a tool that has been used for a wide range of purposes (see section III) in foreign countries and , we think, could be useful in the national setting. Such is the motivation behind our research.

V. Project scope

There are two main approaches to studying the TS:

- **Curve-fitting and interpolation** try to estimate unobserved yields from cross-sectional data.
- **Dynamic models** try to capture the factors that drive yield changes using panel data.

This project focuses on dynamic models, as curve-fitting and interpolation methods are already commonly used in Colombia.

Out of the great variety of dynamic models that have been proposed for describing the TS, this project will work with ATSMs. This class of models are preferred due to their tractability and consistency with no-arbitrage. We will initially work with specific ATSMs whose closed-form likelihood expansions are provided by Ait-Sahalia[3].

As it was discussed in section III and IV, ATSMs have a great range of possible applications. We will only focus in yield forecasts given the timescale of the project and due to the fact that it is our first time working with such

[2]The no-arbitrage hypothesis states that no riskless profit can be made from diferences in prices between markets.

[3]Yacine Ait-Sahalia's Research Page: Closed-Form MLE for Diffusions. http://www.princeton.edu/~yacine/closedformmle.htm

class of models. Derivative pricing and term structure decomposition are left as possible future work.

VI. Methodology

The first step towards model implementation is to obtain the data that will be used for parameter estimation and testing. As the Colombian market is still in its development phase, it is not possible to obtain observations of zero-coupon yields for fixed maturities over long time periods. We will therefore use the Nelson-Siegel model to approximate yield curves. The curves' parameters will be those published daily by *Infoval* in the time period between 1/1/2009 and 1/19/2014.

Afterwards, this data will be used to replicate basic results and exercises from the literature. Those available from Carlo A. Favero[4] are a good starting point. This phase's purpose is to gain insight into the computational implementation and calibration of the models.

We will then use the likelihood expansions provided by Ait-Sahalia[5] to estimate various ATSMs by maximum likelihood, using our yield curve data. We will follow the methodology proposed by Ait-Sahalia and Kimmel[6], using a number of observed yields equal to twice the number of factors in the model (N). The maturities of the selected yields will range from one month to ten years.

Once the models and their estimation procedures are implemented, we will test the models' accuracy in forecasting the yield curve. A reduced sample will be used for estimation and one-period out-of-sample forecasts will be produced. The mean squared forecast error (MSFE) will then be calculated for each maturity and the models will be rated on this basis.

VII. Activity schedule

An estimated schedule of the different phases of the project is presented in table I. As the project is enclosed within the "Research practice II" course from the Mathematical Engineering program, various documents and presentations must be elaborated. The (current) deadlines for these reports and presentations are shown in table II.

VIII. Budget

The project won't require any financing.

[4]Teaching materials: Econometric methods for finance and macroeconomics, Models for the term structure. Bocconi University. http://didattica.unibocconi.eu/myigier/doc.php?idDoc=6312&IdUte=48917&idr=1754&Tipo=m&lingua=eng#2.

[5]See footnote 3.

Table I
Project schedule

Activity / Objective to be met	Time range
Literature review	Jan. 26 - Feb. 28
Implementation and estimation of various models and replication of results from the literature.	Mar 1 - Mar 31.
Testing of models' performance in forecasting the Colombian yield curve.	Apr 1 - May 28

Table II
Course deadlines

Report / Presentation	Deadline
Project proposal report	Feb. 13th.
Project proposal presentation	4th academic week.
Oral progress report	10th academic week.
Final report	May 29th.
Final presentation	19th academic week.

These dates were taken from the course's website: http://www1.eafit.edu.co/asr/courses/research-practises-me/index.html

IX. Intellectual property

In accordance with Universidad EAFIT's intellectual property ruling [17], the patrimonial rights over all the academic products resulting from this project will belong to:

- Mateo Velásquez Giraldo
- Diego Alexander Restrepo Tobón
- Universidad EAFIT

The ruling also states that utilities obtained through commercialization of any product of the project must be divided in the following proportions:

- Mateo Velásquez Giraldo: 25%.
- Diego Alexander Restrepo Tobón: 20%.
- Universidad EAFIT: 55%.

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