Affine term structure models: forecasting the Colombian yield curve

Mateo Velásquez-Giraldo
Mathematical Engineering
EAFIT University

Diego Alexander Restrepo-Tobón
Tutor, Department of Finance
EAFIT University

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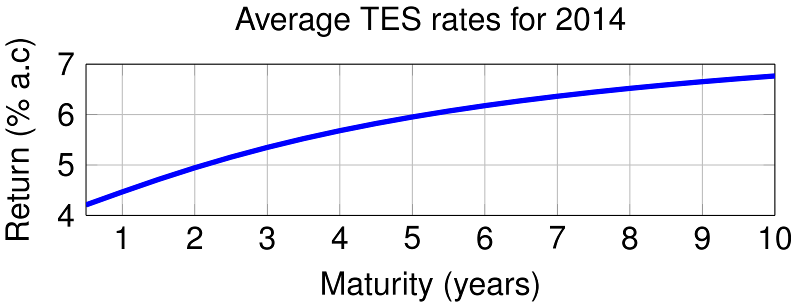
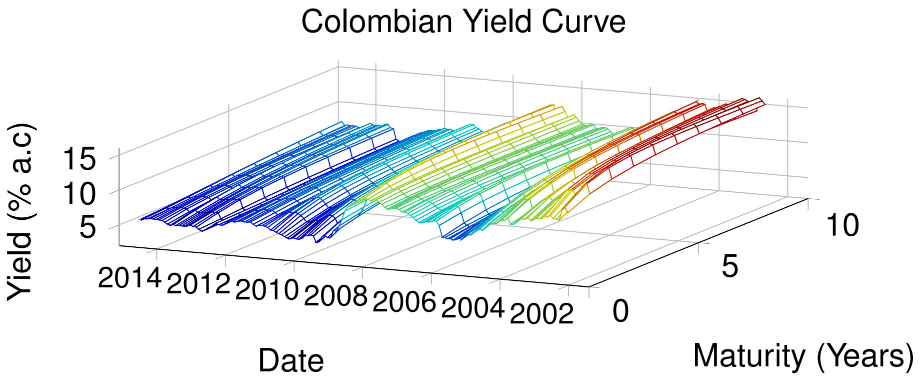
Interest rates

- Not constant!
- Vary with *time* (t) and *maturity* (τ).
- Higher maturities have higher expected returns.



The term structure of interest rates

- Relationship between interest rates and their maturity.
- Dynamic and (mostly) not observable.



Why study the TS?

Understanding the term structure can be useful for:

- Assessing risk.
- Properly discounting cashflows.
- Aiding decisions from investors and policymakers.
- Pricing derivatives.
- Extracting information about the state of the economy and the financial markets.

Affine term structure models

- Model yields over time and maturity $\gamma_\tau(t)$ as affine functions of a latent state vector $X(t)$:

$$\gamma_\tau(t) = A(\tau) + B(\tau)^\top X(t)$$

$$r = \lim_{\tau \to 0} \gamma_\tau = \delta_0 + \delta_1^\top X(t)$$

- $X(t)$ captures changes over **time**.
- $A(\tau)$ and $B(\tau)$ change over **maturities**.
- If N is the number of factors in $X(t)$, $A(\tau) \in \mathbb{R}$ and $B(\tau) \in \mathbb{R}^N$.

State dynamics

- Under the risk-neutral measure Q:

$$dX(t) = \widetilde{K}\left(\widetilde{\Theta} - X(t)\right)dt + \Sigma\sqrt{S(t)}d\widetilde{W}t$$

$$[S(t)]_{i,i} = \alpha_i + \beta_i^{\top}X(t)$$

- The 'real' dynamics are found specifying the price of risk. We use $\Lambda(t) = \sqrt{S(t)}\lambda$, $\lambda \in \mathbb{R}^N$.
- We consider 1-3 factor models.

$A(\tau)$ and $B(\tau)$

- $A(\tau)$ and $B(\tau)$ are obtained by solving the ODE system:

$$a'(\tau) = -\delta_0 + b(\tau)^\top \widetilde{K}\widetilde{\Theta} + \frac{1}{2}\sum_{i=1}^{N}[b(\tau)^\top \Sigma]_i^2 \alpha_i$$

$$b'(\tau) = \delta_1 - \widetilde{K}^\top b(\tau) + \frac{1}{2}\sum_{i=1}^{N}[b(\tau)^\top \Sigma]_i^2 \beta_i$$

- $a(0) = 0$, $b(0) = \vec{0}$, $A(\tau) = -a(\tau)/\tau$ and $B(\tau) = -b(\tau)/\tau$.
- They grant consistency with the **no arbitrage hypothesis** [Duffie and Kan, 1996].

Data

- Nelson-Siegel curves published by *Infovalmer*.
- Time period: Aug.2002-Mar.2015
- Daily observations (3051 bursatile days).
- 2000 days used for model estimation.

Notation

We denote models as:

$$A_M(N)$$

Where:

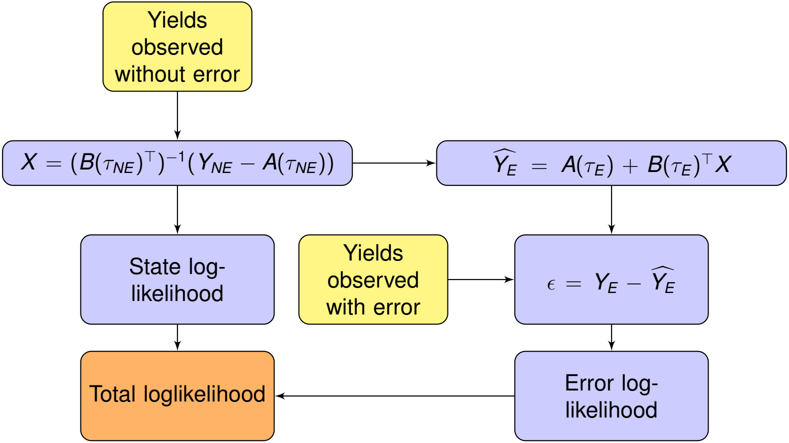
- N is the number of factors in the model (1, 2 or 3).
- $M \leq N$ is the number of factors which affect volatility.

Assumptions

- For an N factor model, we assume N yields are observed without error:
    - 1 year yield for $N = 1$.
    - 1 and 10 year yields for $N = 2$.
    - 1, 5 and 10 year yields $N = 3$.
- 3, 6 and 9 year yields are assumed to be observed with Gaussian errors for all models.

The log-likelihood function I

For a given set of parameters:



The log-likelihood function II

- This methodology is proposed in [Ait-Sahalia and Kimmel, 2010].
- The state likelihood is obtained through the approximations in [Ait-Sahalia, 2008].
- Optimization is complicated. For each set of parameters:
    - Non linear restrictions are checked.
    - A system of differential equations is solved.
    - Time series of states and yields are obtained.

Differential evolution

```
Data: # Generations:ng, Population size:np, F ∈ [0,2],
      CR ∈ [0,1]
P1 ← Random initial population
for i = 1 to ng do
    P0 ← P1
    for j = 1 to np do
        {a, b, c} ← random individuals from P0
        V ← a + F * (b − c)
        for k = 1 to #Params do
            if rand ≤ CR then
                U_k ← V_k
            else
                U_k ← P0(j)_k
            end
        end
        if fobj(U) ≤ P0(j) then
            P1(j) ← U
        end
    end
end
```

Simulated state tests

Simulate a trajectory and try to obtain its parameters.

		fminsearch(...)		Differential Evolution	
Model	**Loglikelihood with real params**	**Loglikelihood**	**Mean parameter relative error**	**Loglikelihood**	**Mean parameter relative error**
$A_0(1)$	4049	4049	1%	4049	1%
$A_1(1)$	4967, 6	4967, 6	60348%	6.1×10^{20}	109710%
$A_0(2)$	8023	8023, 5	43%	8023, 5	43%
$A_1(2)$	6440, 5	6449, 9	82%	6449, 9	80%
$A_2(2)$	5185, 7	5185, 3	151%	5187, 3	26%
$A_0(3)$	12181	12186	213%	12186	203%
$A_1(3)$	12533	12534	228%	$1,4 \times 10^{35}$	825%
$A_2(3)$	12885	12830	914%	12891	140%
$A_3(3)$	6951, 9	6954, 6	400%	6960, 2	634%

- Identification problems.
- Enormous, wrong log-likelihoods.

Estimation results

Three types of results were obtained:

$A_2(2)$ and $A_3(3)$ No feasible solutions found.

$A_1(3)$ and $A_2(3)$ Error in state-log likelihood.

$A_0(1)$, $A_1(1)$, $A_0(2)$, $A_1(2)$ and $A_0(3)$ Successful.

Error in state log-likelihood

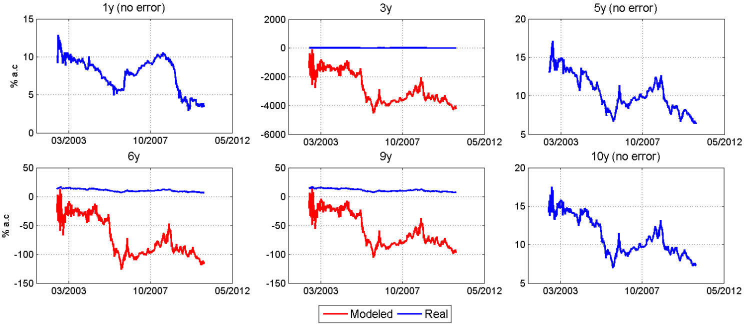


Figure 1: $A_1(3)$ results.

The wrong state log-likelihood makes errors irrelevant.

$A_0(1)$ model

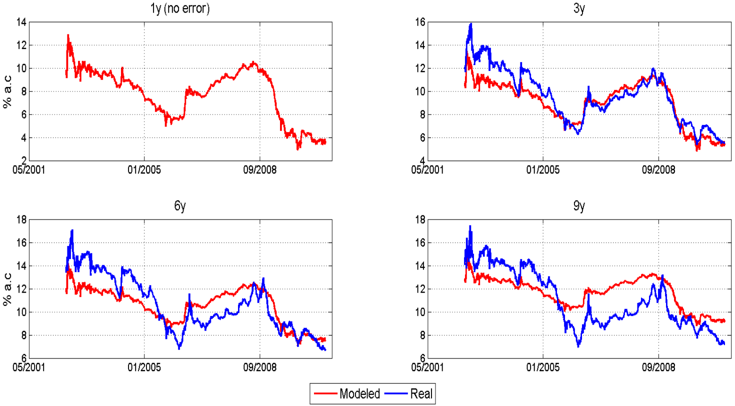


Figure 2: $A_0(1)$ results.

$A_1(1)$ model

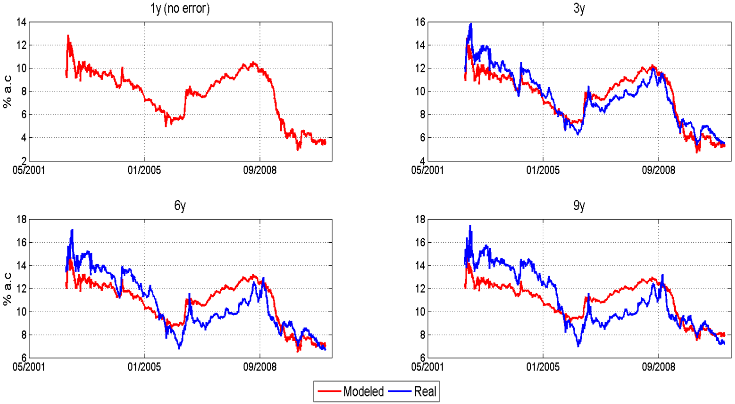


Figure 3: $A_1(1)$ results.

$A_0(2)$ model

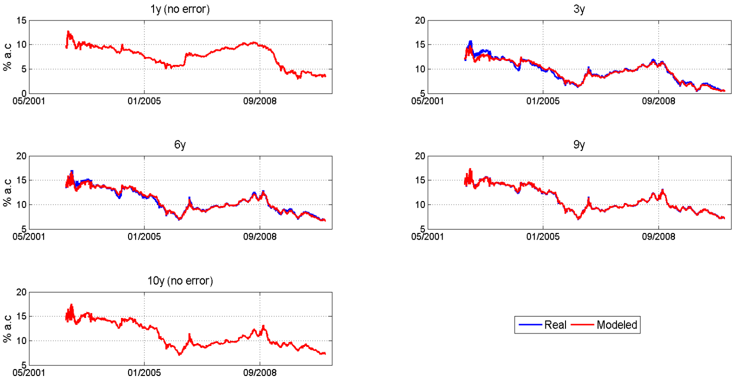


Figure 4: $A_0(2)$ results.

$A_1(2)$ model

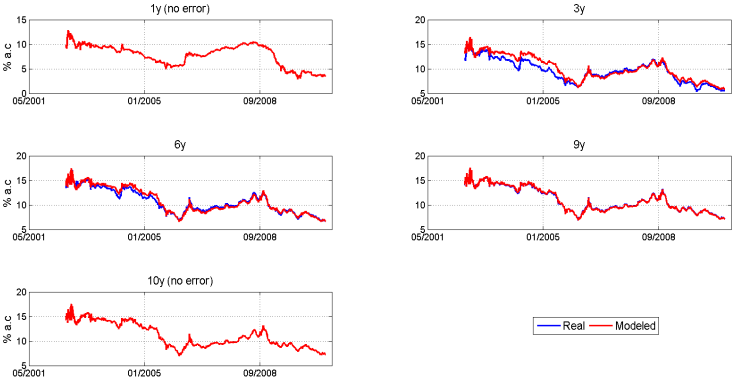


Figure 5: $A_1(2)$ results.

$A_0(3)$ model

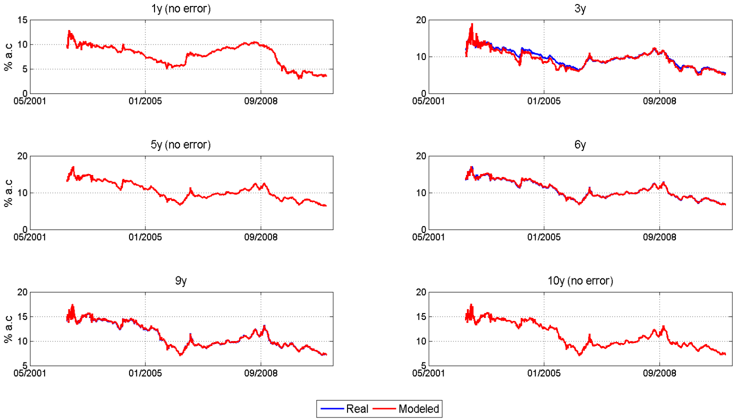


Figure 6: $A_0(3)$ results.

Modelling more yields with $A_0(3)$

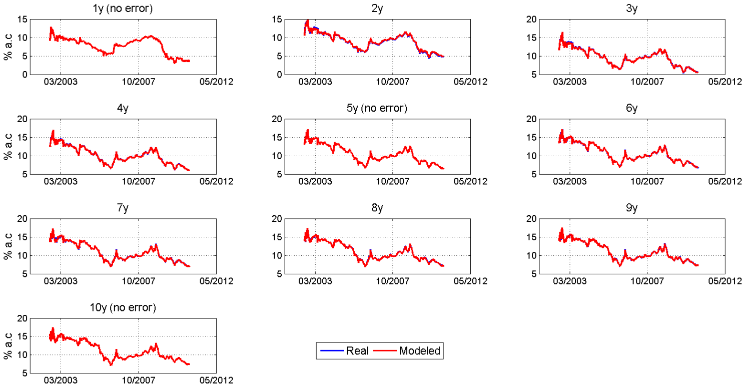


Figure 7: $A_0(3)$ results with 7 yields observed with error.

Conclusions

- An estimation procedure for five ATSM's has been implemented successfully with Colombian data.
- The $A_0(3)$ model can represent a high number of Colombian yields very closely.
- The Colombian term structure can be adjusted to models consistent with no arbitrage.

Future work

- Find the cause of the problem with the state log-likelihood approximations.
- Test the models' forecast accuracy.
- Confidence analysis of estimated parameters.
- Applications of the models, e.g., estimation of risk premiums and derivative pricing.

First forecast results

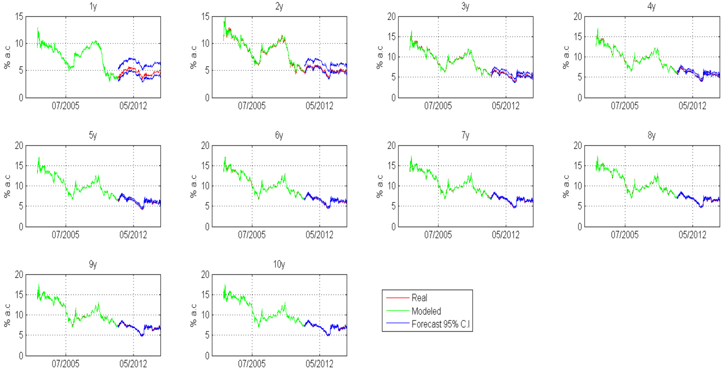


Figure 8: Out of sample forecasts with the $A_0(3)$ model and 5-day horizons.

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Thanks for your attention!