Ordinals and Typed Lambda Calculus

Ordinal Numbers

Andrés Sicard-Ramírez

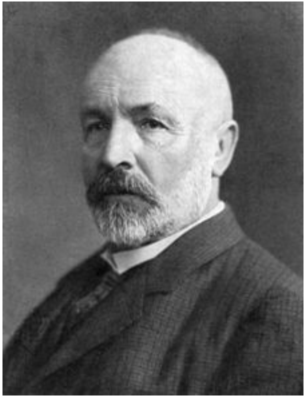
Universidad EAFIT

Semester 2018-2

Ordinal and Cardinal Numbers

Georg Cantor (1845 – 1918)*

Cantor around 1870



*Figures source: Wikipedia.

Ordinal and Cardinal Numbers

Generalisation of natural numbers via two abstractions

Cantor's definitions of ordinal and cardinal numbers were 'definitions by abstraction' [Ivorra Castillo 2017, pp. 293-4]:

> *Aquí [Cantor (1915) 1955] define Cantor el 'ordinal' de un conjunto ordenado como el concepto al que llegamos cuando hacemos abstracción de la naturaleza de sus elementos y conservamos únicamente su ordenación, de modo que dos conjuntos tienen el mismo ordinal si y sólo si sus elementos pueden ponerse en correspondencia biunívoca conservando el orden. Por otra parte, el 'cardinal' de un conjunto es el concepto al que llegamos cuando hacemos abstracción de la naturaleza de sus elementos, así como de toda posible ordenación.*

Ordinal Numbers Definition

Some definitions

- Cantor (1883). Equivalence class of well-ordered sets.
- von Neumann (1923). Canonical well-ordered sets.
- von Neumann (1928). Transitive sets well-ordered by the membership relation.
- Robinson (1937). Sets satisfying some properties without using the theory of ordered sets.

Cantor (1883): Ordinals as Equivalence Class of Well-Ordered Sets

Definition

A **structure** is a pair (A, R) consisting of a set A and a binary relation R on A.*

*See, e.g. [Enderton 1977].

Cantor (1883): Ordinals as Equivalence Class of Well-Ordered Sets

Definition

A **structure** is a pair (A, R) consisting of a set A and a binary relation R on A.*

Definition

Let (A, R) be a structure where R is a relation of order. The Cantor **order-type** of (A, R) is the collection of the **similar** (i.e. order-isomorphic) structures to it.†

*See, e.g. [Enderton 1977].

†Some authors define the order-types on totally ordered sets (see, e.g. [Mendelson (1964) 2015, p. 249]) or on well-ordered sets (see, e.g. [Hrbacek and Jech (1978) 1999, p. 113]). The collection of similar structures is not a set but a class. A generalisation of the above definitions is the following. A **relational**/**algebraic** structure is a set and various relations/operations on it. In general, a **structure** is a set and various relations and operations on it (see, e.g. [Hrbacek and Jech (1978) 1999]). The structures are called 'isomorphic' instead of 'similar' when working with general structures.

Cantor (1883): Ordinals as Equivalence Class of Well-Ordered Sets

Definition

An **ordinal number** is the order-type of a well-ordered set.*

Remark

Note the ‘definition by abstraction’ in the previous definition.

*Cantor’s definition [Cantor (1883) 1976]. See, e.g. [van Heijenoort 1967, p. 346], [Suppes (1960) 1972, p. 129] and [Moore 1982, p. 52].

Cantor (1883): Ordinals as Equivalence Class of Well-Ordered Sets

Definition

An **ordinal number** is the order-type of a well-ordered set.*

Remark

Note the 'definition by abstraction' in the previous definition.

Drawback

The ordinal numbers are not sets but classes.

*Cantor's definition [Cantor (1883) 1976]. See, e.g. [van Heijenoort 1967, p. 346], [Suppes (1960) 1972, p. 129] and [Moore 1982, p. 52].

Cantor (1883): Ordinals as Equivalence Class of Well-Ordered Sets

Definition

An **ordinal number** is the order-type of a well-ordered set.*

Remark

Note the 'definition by abstraction' in the previous definition.

Drawback

The ordinal numbers are not sets but classes.

Discussion

Can we define a data type for representing ordinal numbers from the previous definition?

*Cantor's definition [Cantor (1883) 1976]. See, e.g. [van Heijenoort 1967, p. 346], [Suppes (1960) 1972, p. 129] and [Moore 1982, p. 52].

von Neumann (1923): Ordinals as Canonical Well-Ordered Sets

Motivation

To define the ordinal numbers as sets.

von Neumann (1923): Ordinals as Canonical Well-Ordered Sets

Motivation

To define the ordinal numbers as sets.

von Neumann's informal definition

von Neumann [(1923) 1967] initially described the ordinals numbers by:

i) Every ordinal is the set of ordinals that precede it.

ii) The first ordinal is the empty set.

von Neumann (1923): Ordinals as Canonical Well-Ordered Sets

Motivation

To define the ordinal numbers as sets.

von Neumann's informal definition

von Neumann [(1923) 1967] initially described the ordinals numbers by:

i) Every ordinal is the set of ordinals that precede it.

ii) The first ordinal is the empty set.

Definition

Let a be a set. The **successor** of a is defined by

$$\operatorname{succ} a := a \cup \{a\}.$$

von Neumann (1923): Ordinals as Canonical Well-Ordered Sets

Example

Some von Neumann ordinals.

$$
\begin{aligned}
0 &:= \emptyset, \\
1 &:= \operatorname{succ} 0 = \{0\} = \{\emptyset\}, \\
2 &:= \operatorname{succ} 1 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}, \\
&\ \ \dots \\
\omega &:= \{0, 1, 2, \dots\}.
\end{aligned}
$$

von Neumann (1923): Ordinals as Canonical Well-Ordered Sets

Example

Some von Neumann ordinals.

$$\begin{aligned}
0 &\coloneqq \emptyset, \\
1 &\coloneqq \operatorname{succ} 0 = \{0\} = \{\emptyset\}, \\
2 &\coloneqq \operatorname{succ} 1 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}, \\
&\dots \\
\omega &\coloneqq \{0, 1, 2, \dots\}.
\end{aligned}$$

Note that

$$0 \in 1 \in 2 \in \cdots \quad \text{and} \quad 0 \subseteq 1 \subseteq 2 \subseteq \cdots .$$

von Neumann (1923): Ordinals as Canonical Well-Ordered Sets

Example

Some von Neumann ordinals.

$$
\begin{aligned}
\omega &\coloneqq \{0, 1, 2, \dots\}, \\
\omega + 1 &\coloneqq \operatorname{succ} \omega = \{0, 1, 2, \dots, \omega\}, \\
\omega + 2 &\coloneqq \operatorname{succ} (\omega + 1) = \{0, 1, 2, \dots, \omega, \omega + 1\}, \\
&\dots \\
\omega \cdot 2 &\coloneqq \omega + \omega = \{0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots\}.
\end{aligned}
$$

von Neumann (1923): Ordinals as Canonical Well-Ordered Sets

Example

Some von Neumann ordinals.

$$
\begin{aligned}
\omega &:= \{0, 1, 2, \dots\}, \\
\omega + 1 &:= \operatorname{succ} \omega = \{0, 1, 2, \dots, \omega\}, \\
\omega + 2 &:= \operatorname{succ} (\omega + 1) = \{0, 1, 2, \dots, \omega, \omega + 1\}, \\
&\dots \\
\omega \cdot 2 &:= \omega + \omega = \{0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots\}.
\end{aligned}
$$

Note also that

$$\omega \in \omega + 1 \in \omega + 2 \in \cdots \quad \text{and} \quad \omega \subseteq \omega + 1 \subseteq \omega + 2 \subseteq \cdots .$$

von Neumann (1923): Ordinals as Canonical Well-Ordered Sets

Definition

Let $(A, \prec)$ be a well-ordered set and let $a \in A$. The **initial $\prec$-segment of A generated by a** is the set of all elements of A that strictly precede a, that is,

$$\operatorname{seg} a \coloneqq \{\, x \in A \mid x \prec a \,\}.$$

von Neumann (1923): Ordinals as Canonical Well-Ordered Sets

Definition

Let $(A, \prec)$ be a well-ordered set and let $a \in A$. The **initial $\prec$-segment of A generated by a** is the set of all elements of A that strictly precede a, that is,

$$\operatorname{seg} a := \{ x \in A \mid x \prec a \}.$$

von Neumann's (1923) definition

'A set x is an ordinal number iff there exists a well-ordering R on x such that every element of x is equal to its corresponding initial R-segment.' [Dunik 1966, p. 13]

von Neumann (1923): Ordinals as Canonical Well-Ordered Sets

Definition

Let $(A, \prec)$ be a well-ordered set and let $a \in A$. The **initial $\prec$-segment of A generated by a** is the set of all elements of A that strictly precede a, that is,

$$\operatorname{seg} a \coloneqq \{\, x \in A \mid x \prec a \,\}.$$

von Neumann's (1923) definition

'A set x is an ordinal number iff there exists a well-ordering R on x such that every element of x is equal to its corresponding initial R-segment.' [Dunik 1966, p. 13]

Example

For the ordinal number $3 = \{0, 1, 2\}$, the elements $0, 1$ and 2 are equals to the initial ϵ-segments $\operatorname{seg} 0$, $\operatorname{seg} 1$ and $\operatorname{seg} 2$, respectively.

von Neumann (1923): Ordinals as Canonical Well-Ordered Sets

Drawback

The ordinal numbers are sets but their definition is based on well-orderings.

von Neumann (1923): Ordinals as Canonical Well-Ordered Sets

Drawback

The ordinal numbers are sets but their definition is based on well-orderings.

Discussion

Can we define a data type for representing ordinal numbers from the previous definition?

Let's Count

Fasting

$$0, 1, 2, \ldots, \omega, \omega+1, \omega+2, \ldots, w\cdot 2, \ldots, \omega\cdot 3, \ldots,$$

$$w^2, \ldots, \omega^3, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots, \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}, \ldots,$$

Let's Count

A little slower

$0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega \cdot 2,$

Let's Count

A little slower

$0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega \cdot 2,$

$\omega \cdot 2 + 1, \omega \cdot 2 + 2, \ldots, \omega \cdot 3, \ldots, \omega \cdot 4, \ldots, \omega \cdot \omega = \omega^2,$

Let's Count

A little slower

$0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega \cdot 2,$

$\omega \cdot 2 + 1, \omega \cdot 2 + 2, \ldots, \omega \cdot 3, \ldots, \omega \cdot 4, \ldots, \omega \cdot \omega = \omega^2,$

$\omega^2 + 1, \omega^2 + 2, \ldots, \omega^2 + \omega, \omega^2 + \omega + 1, \omega^2 + \omega + 2, \ldots, \omega^2 + \omega \cdot 2, \ldots,$

Let's Count

A little slower

$0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega \cdot 2,$

$\omega \cdot 2 + 1, \omega \cdot 2 + 2, \ldots, \omega \cdot 3, \ldots, \omega \cdot 4, \ldots, \omega \cdot \omega = \omega^2,$

$\omega^2 + 1, \omega^2 + 2, \ldots, \omega^2 + \omega, \omega^2 + \omega + 1, \omega^2 + \omega + 2, \ldots, \omega^2 + \omega \cdot 2, \ldots,$

$\omega^2 + \omega \cdot 3, \ldots, \omega^2 \cdot 2, \ldots, \omega^2 \cdot 3, \ldots, \omega^3, \ldots, \omega^4, \ldots, \omega^\omega,$

Let's Count

A little slower

$0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega \cdot 2,$

$\omega \cdot 2 + 1, \omega \cdot 2 + 2, \ldots, \omega \cdot 3, \ldots, \omega \cdot 4, \ldots, \omega \cdot \omega = \omega^2,$

$\omega^2 + 1, \omega^2 + 2, \ldots, \omega^2 + \omega, \omega^2 + \omega + 1, \omega^2 + \omega + 2, \ldots, \omega^2 + \omega \cdot 2, \ldots,$

$\omega^2 + \omega \cdot 3, \ldots, \omega^2 \cdot 2, \ldots, \omega^2 \cdot 3, \ldots, \omega^3, \ldots, \omega^4, \ldots, \omega^\omega,$

$\omega^\omega + 1, \omega^\omega + 2, \ldots, \omega^{\omega \cdot 2}, \ldots, \omega^{\omega^2}, \ldots, \omega^{\omega^\omega}, \ldots, \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}, \ldots,$

von Neuman (1928): Ordinals as Transitive Sets Well-Ordered by the Membership Relation

Definition

A set A is **well-ordered by the membership relation** iff the relation

$$\epsilon_A := \{ \langle x, y \rangle \in A \times A \mid x \in y \}$$

is a well-ordering on A.

von Neuman (1928): Ordinals as Transitive Sets Well-Ordered by the Membership Relation

Definition

A set A is **well-ordered by the membership relation** iff the relation

$$\epsilon_A \coloneqq \{\, \langle x, y \rangle \in A \times A \mid x \in y \,\}$$

is a well-ordering on A.

Definition

A set S is **transitive** iff every element of S is a subset of S.*

*See, e.g. [Hrbacek and Jech (1978) 1999, Ch. 6, Def. 2.1].

von Neuman (1928): Ordinals as Transitive Sets Well-Ordered by the Membership Relation

Definition

A set α is an **ordinal number** iff *

- the set is transitive and
- the set is well-ordered by the membership relation.

*von Neumann's definition [von Neumann 1928]. See, e.g. [Rubin 1967, p. 176] and [Jech 1971, p. 7]. This is the definition used by [Hrbacek and Jech (1978) 1999].

Robinson (1937): Ordinals as Sets Satisfying Some Properties

Motivation

The theory of ordinal numbers can be constructed without using the theory of ordered sets.

Robinson (1937): Ordinals as Sets Satisfying Some Properties

Motivation

The theory of ordinal numbers can be constructed without using the theory of ordered sets.

Definition

A set A is **transitive** iff

$$\forall a\,(a \in A \to a \subseteq A),$$

or equivalently, iff

$$\forall x\,\forall a\,(x \in a \in A \to x \in A).$$

Robinson (1937): Ordinals as Sets Satisfying Some Properties

Definition

A set A is **connected under ϵ** iff

$$\forall x \, \forall y \, (x, y \in A \to x \in y \lor x = y \lor y \in x).$$

Robinson (1937): Ordinals as Sets Satisfying Some Properties

Definition

A set A is **connected under ϵ** iff

$$\forall x \, \forall y \, (x, y \in A \to x \in y \lor x = y \lor y \in x).$$

Definition

A set α is an **ordinal number** iff [Robinson 1937]*

- the set is transitive and
- the set is connected under ϵ.

*Drake [1974, p. 24] introduces this definition under Zermelo-Fraenkel set theory. See, also, [Dunik 1966, p. 14] and [van Heijenoort 1967, p. 348]. It is also the definition used by the Mizar Mathematical Library [Bancerek 1990].

Robinson (1937): Ordinals as Sets Satisfying Some Properties

Discussion

Can we define a data type for representing ordinal numbers from the previous definition?

Properties of Ordinals

Remark

Recall that a set α is an ordinal number iff

- the set is transitive and
- the set is well-ordered by the membership relation.

Some properties

See slides for the subject 'Ordinal Numbers' in `http://www1.eafit.edu.co/asr/courses/set-theory-cm0832/`.

Properties of Ordinals

Definition

There are three sorts of ordinal numbers:* An ordinal number α is

i) the **zero ordinal**,

ii) a **successor ordinal** iff α has an immediate predecessor, i.e. $\alpha = \operatorname{succ} \beta$ for some ordinal β, or

iii) a **limit ordinal** iff α is non-zero and α has no an immediate predecessor.†

*See, e.g. [Rogers (1967) 1992, p. 220] and [Enderton 1977, p. 203].

†Since zero has no an immediate predecessor, some authors include it as a limit ordinal, see, e.g. [Hrbacek and Jech (1978) 1999, p. 108].

An Ordinal Numbers Class

Remark

From previous theorems we know that:

i) zero is an ordinal,

ii) if α is an ordinal, then $\operatorname{succ} \alpha$ is an ordinal and

iii) if A is a set of ordinals, then $\bigcup A$ is an ordinal.

An Ordinal Numbers Class

Definition

Rubin [1967, pp. 175-6] points out that under an appropriate (but not explicitly mentioned) axiomatic, the **class of ordinal numbers**, denoted by On, could be defined by*

$$\frac{}{\emptyset \in \mathsf{On}} \qquad \frac{\alpha \in \mathsf{On}}{\operatorname{succ} \alpha \in \mathsf{On}} \qquad \frac{A \subseteq \mathsf{On}}{\bigcup A \in \mathsf{On}}$$

*See, also, [Crossley 1969, p. 11].

An Ordinal Numbers Class

Definition

Rubin [1967, pp. 175-6] points out that under an appropriate (but not explicitly mentioned) axiomatic, the **class of ordinal numbers**, denoted by On, could be defined by*

$$\frac{}{\emptyset \in \mathsf{On}} \qquad \frac{\alpha \in \mathsf{On}}{\operatorname{succ} \alpha \in \mathsf{On}} \qquad \frac{A \subseteq \mathsf{On}}{\bigcup A \in \mathsf{On}}$$

Discussion

Can we inductively define classes? Suggesting reading: Aczel [1977] and Curi [2018].

*See, also, [Crossley 1969, p. 11].

An Ordinal Numbers Class

Definition

Phillips [1992, pp. 158-9] introduces the **ordinals** via the following inductive definition:

- Zero is an ordinal.
- If α is an ordinal, then $\operatorname{succ} \alpha$ is an ordinal.
- If A is a downwards-closed segment of ordinals (i.e. if $\alpha \in A$ and $\beta < \alpha$, then $\beta \in A$) with no greatest element, then $\sup A$ is an ordinal.

Discussion

However, the author points out the following flaws of above definition:

> *In order to understand 'downwards-closed' we have to define the ordering on ordinals. This will have to be defined simultaneously with the ordinals. Furthermore the inductive clause for limits use an infinite set A. So the definition is not finitary.*

Transfinite Induction and Recursion

Theorem (Transfinite induction principle (one version))

Let $\varphi(x)$ a propositional function. Assume that

i) $\varphi(0)$.

ii) For all ordinals α, $\varphi(\alpha)$ implies $\varphi(\operatorname{succ}\alpha)$.

iii) For all limit ordinals, if $\varphi(\beta)$ for all $\beta < \alpha$, then $\varphi(\alpha)$.

Then $\varphi(\alpha)$ for all ordinals α.*

*See, e.g. [Hrbacek and Jech (1978) 1999, p. 115].

Transfinite Induction and Recursion

Theorem (Transfinite recursion theorem)

TODO.

Ordinal Arithmetic

Definition

We define the **addition** of ordinals by transfinite recursion. For any ordinals α and β:

$$\begin{aligned}
\alpha + 0 &= \alpha, \\
\alpha + (\operatorname{succ}\beta) &= \operatorname{succ}(\alpha + \beta), \\
\alpha + \beta &= \sup\{\, \alpha + \delta \mid \delta < \beta \,\}, \text{ if } \beta \text{ is a limit ordinal.}
\end{aligned}$$

Example

If $\alpha = \beta = \omega$, then

$$\omega + \omega = \sup\{\, \omega + n \mid n < \omega \,\}.$$

Ordinal Arithmetic

Definition

We define the **multiplication** of ordinals by transfinite recursion. For any ordinals α and β:

$$\begin{aligned}
\alpha \cdot 0 &= 0, \\
\alpha \cdot (\operatorname{succ} \beta) &= \alpha \cdot \beta + \alpha, \\
\alpha \cdot \beta &= \sup \{\, \alpha \cdot \delta \mid \delta < \beta \,\}, \text{ if } \beta \text{ is a limit ordinal.}
\end{aligned}$$

Ordinal Arithmetic

Definition

We define the **exponentiation** of ordinals by transfinite recursion. For any ordinals α and β:

$$\begin{aligned} \alpha^0 &= 1, \\ \alpha^{(\operatorname{succ}\beta)} &= \alpha^\beta \cdot \alpha, \\ \alpha^\beta &= \sup\left\{ \alpha^\delta \;\middle|\; \delta < \beta \right\}, \text{ if } \beta \text{ is a limit ordinal.} \end{aligned}$$

Cantor Normal Formal

Remark

Recall that any positive integer has a decimal representation. For example,

$$562 = 5 * 10^2 + 6 * 10^1 + 2 * 10^0.$$

Cantor Normal Formal

Remark

Recall that any positive integer has a decimal representation. For example,

$$562 = 5 * 10^2 + 6 * 10^1 + 2 * 10^0.$$

Theorem

Let b be an integer greater than 1. Every integer $x > 0$ can be uniquely represented as

$$x = x_n * b^n + x_{n-1} * b^{n-1} + \cdots + x_1 * b^1 + x_0 * b^0,$$

where $x_n \neq 0$ and every x_i is a non-negative integer less that b.* This representation is the **representation of x to the base b**.

*See, e.g. [Andrews 1971, Theorem 1-3, p. 8].

Cantor Normal Formal

Example (Decimal representation)

Every integer $x > 0$ can be uniquely represented as

$$x = x_n * 10^n + x_{n-1} * 10^{n-1} + \cdots + x_1 * 10^1 + x_0 * 10^0,$$

where $x_n \neq 0$, and $0 \leq x_i < 10$, for every x_i.

Cantor Normal Formal

Theorem (Cantor normal form (one version))

Every ordinal $\alpha > 0$ can be uniquely represented as

$$\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_n} \cdot k_n,$$

where $\beta_1 > \beta_2 > \cdots > \beta_n$ are ordinals and every k_i is a non-zero finite ordinal.

Cantor Normal Formal

Theorem (Cantor normal form (other version))

By setting all the non-zero finite ordinals k_i equal to 1 and allowing the exponents to be equal, Cantor normal form can be rewriting by:

Every ordinal $\alpha > 0$ can be uniquely represented as

$$\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_n},$$

where $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ are ordinals.

Natural Operations

Definition

Let α and β be ordinals. We can represent them by

$$\alpha = \omega^{\delta_1} \cdot k_1 + \omega^{\delta_2} \cdot k_2 + \cdots + \omega^{\delta_n} \cdot k_n,$$
$$\beta = \omega^{\delta_1} \cdot j_1 + \omega^{\delta_2} \cdot j_2 + \cdots + \omega^{\delta_n} \cdot j_n,$$

where $\delta_1 > \delta_2 > \cdots > \delta_n$ are ordinals and every k_i and j_i is a finite ordinal.

The **natural sum** of α and β is defined by*

$$\alpha \mathbin{\#} \beta \coloneqq \omega^{\delta_1} \cdot (k_1 + j_1) + \omega^{\delta_2} \cdot (k_2 + j_2) + \cdots + \omega^{\delta_n} \cdot (k_n + j_n).$$

*See, e.g. [Sierpiński (1958) 1965, § 28, p. 366].

Natural Operations

Theorem

For any ordinals α and β:*

i) $\alpha \# \beta = \beta \# \alpha$.

ii) $\alpha + \beta \leq \alpha \# \beta$.

*See, e.g. [Sierpiński (1958) 1965, § 28, p. 366].

References

Aczel, Peter (1977). An Introduction to Inductive Definitions. In: Handbook of Mathematical Logic. Ed. by Barwise, Jon. Vol. 90. Studies in Logic and the Foundations of Mathematics. Elsevier. Chap. C.7. DOI: 10.1016/S0049-237X(08)71120-0 (cit. on pp. 39, 40).

Andrews, George E. (1971). Number Theory. Dover Publications (cit. on p. 48).

Bancerek, Grzegorz (1990). The Ordinal Numbers. Formalized Mathematics 1.1, pp. 91–96. URL: http://mizar.org/fm/1990-1/fm1-1.html (cit. on p. 34).

Cantor, Georg [1915] (1955). Contributions to the Founding of the Theory of Transfinite Numbers. Translation of *Beiträge zur Begründung der transfiniten Mengenlehre*. Mathematische Annalen, 1895, vol. 46.2, pp. 481–512 and 1897, vol. 49.2, pp. 207–246. Translated, and provided with an introduction and notes, by Philip E. B. Jourdain. Dover Publications (cit. on p. 3).

— [1883] (1976). Foundations of a Theory General of Manifols. A Mathematical-Philosophical Study in the Theory of the Infinite. The Campaigner 9.1–2. Translated by Uwe Parpart, pp. 69–96 (cit. on pp. 7–9).

Crossley, John N. (1969). Constructive Order Types. Vol. 56. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Company (cit. on pp. 39, 40).

References

Curi, Giovanni (2018). Abstract Inducive and Co-Inductive Definitions. The Journal of Symbolic Logic 83.2, pp. 598–616. DOI: 10.1017/jsl.2018.13 (cit. on pp. 39, 40).

Drake, Frank R. (1974). Set Theory. An Introduction to Large Cardinals. Vol. 76. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Company (cit. on p. 34).

Dunik, Peter Anthony (1966). On the Foundations of the Theory of Ordinal Numbers. MA thesis. Department of Mathematicas. University of British Columbia. URL: https://open.library.ubc.ca/cIRcle/collections/ubctheses/831/items/1.0080598 (cit. on pp. 17–19, 34).

Enderton, Herbert B. (1977). Elements of Set Theory. Academic Press (cit. on pp. 5, 6, 37).

Hrbacek, Karel and Jech, Thomas [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on pp. 6, 29, 30, 37, 42).

Ivorra Castillo, Carlos (2017). Lógica y Teoría de Conjuntos. URL: https://www.uv.es/ivorra/ (visited on 27/11/2017) (cit. on p. 3).

Jech, Thomas J. (1971). Lectures in Set Theory with Particular Emphasis on the Method of Forcing. Springer-Verlag (cit. on p. 30).

Mendelson, Elliott [1964] (2015). Introduction to Mathematical Logic. 6th ed. CRC Press (cit. on p. 6).

References

- Moore, Gregory H. (1982). Zermelo's Axiom of Choice. Its Origins, Development, and Influence. Springer-Verlag (cit. on pp. 7–9).
- Phillips, I. C. C. (1992). Recursion Theory. In: Handbook of Logic in Computer Science. Ed. by Abramsky, S., Gabbay, Dov M. and Maibaum, T. S. E. Vol. 1. Clarendon Press, pp. 79–187 (cit. on p. 41).
- Robinson, Raphael M. (1937). The Theory of Classes. A Modification of von Neumann's System. The Journal of Symbolic Logic 2.1, pp. 29–36. DOI: 10.1017/S0022481200039463 (cit. on pp. 33, 34).
- Rogers, Hartley [1967] (1992). Theory of Recursive Functions and Effective Computability. Third printing. MIT Press (cit. on p. 37).
- Rubin, Jean E. (1967). Set Theory for the Mathematician. Holden-Day (cit. on pp. 30, 39, 40).
- Sierpiński, Wacław [1958] (1965). Cardinal and Ordinal Numbers. Second edition revised. Translated from Polish by Janina Smólska. PWN (cit. on pp. 52, 53).
- Suppes, Patrick [1960] (1972). Axiomatic Set Theory. Corrected republication. Dover Publications (cit. on pp. 7–9).

References

v. Neumann, J. (1928). Die Axiomatisierung der Mengenlehre. Mathematische Zeitschrift 27, pp. 669–752. DOI: 10.1007/BF01171122 (cit. on p. 30).

van Heijenoort, Jean, ed. (1967). From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931. Source Books in the History of the Sciences. Harvard University Press (cit. on pp. 7–9, 34).

von Neumann, John [1923] (1967). On the Introduction of Transfinite Numbers. In: From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931. Ed. by van Heijenoort, Jean. Source Books in the History of the Sciences. Harvard University Press, pp. 346–354 (cit. on pp. 10–12).