Ordinals and Typed Lambda Calculus

Lambda Calculus

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Introduction

Alonzo Church (1903 – 1995)*



*Figures sources: History of computers, Wikipedia and MacTutor History of Mathematics.

Introduction

Some remarks

- A formal system invented by Church around 1930s.
- The goal was to use the λ-calculus in the foundation of mathematics.
- Intended for studying functions and recursion.
- Computability model.
- A free-type functional programming language.
- λ-notation (e.g. anonymous functions and currying).

Application, Abstraction and Curryfication

Application

Application of the function M to argument N is denoted by $M\ N$ (juxtaposition).

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Abstraction

'If M is any formula containing the variable x, then $\lambda x[M]$ is a symbol for the function whose values are those given by the formula.' [Church 1932, p. 352]

Application, Abstraction and Curryfication

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Abstraction

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Curryfication

'Adopting a device due to Schönfinkel, we treat a function of two variables as a function of one variable whose values are functions of one variable, and a function of three or more variables similarly.' [Church 1932, p. 352]

Lambda Terms

Definition

Let V be a denumerable set of variables. The set of **$\boldsymbol{\lambda}$-terms**, denoted by Λ, is inductively defined by

$$x \in V \Rightarrow x \in \Lambda \qquad \text{(variable)}$$

$$M, N \in \Lambda \Rightarrow (M\ N) \in \Lambda \qquad \text{(application)}$$

$$M \in \Lambda, x \in V \Rightarrow (\lambda x.M) \in \Lambda \qquad \text{(λ-abstraction)}$$

Lambda Terms

Remark

Usually, the set of λ-terms Λ is defined by the abstract grammar*

$$\begin{aligned} \Lambda \ni t ::= {} & x && \text{(variable)} \\ \mid {} & t\,t && \text{(application)} \\ \mid {} & \lambda x.t && \text{(λ-abstraction)} \end{aligned}$$

*See, e.g. [Pierce 2002].

Lambda Terms

Notation

The symbol '$\equiv$' denotes the syntactic identity.

Conventions

- λ-term variables will be denoted by $x, y, z, \ldots$.
- λ-terms will be denoted by $M, N, P, Q, \ldots$.

Lambda Terms

Conventions and syntactic sugar

- Outermost parentheses are not written.
- Application has higher precedence, that is,

$$\lambda\, x.\, M\; N \coloneqq (\lambda\, x.\, (M\; N)).$$

- Application associates to the left, that is,

$$M\, N_1\, N_2 \ldots N_k \coloneqq (\ldots((M\, N_1)\, N_2) \ldots N_k).$$

- Lambda abstraction associates to the right, that is,

$$\begin{aligned} \lambda\, x_1\, x_2 \ldots x_n.\, M &\coloneqq \lambda\, x_1.\, \lambda\, x_2.\, \ldots \lambda\, x_n.\, M \\ &\coloneqq (\lambda\, x_1.\, (\lambda\, x_2.\, (\ldots (\lambda x_n.M) \ldots))). \end{aligned}$$

Lambda Terms

Example

Using the conventions and syntactic sugar.

$$
\begin{aligned}
&(\lambda\, x\, y\, z.\, x\, z\, (y\, z))\, u\, v\, w \\
&\equiv (\lambda\, x\, y\, z.\, (x\, z)\, (y\, z))\, u\, v\, w && \text{(left-associative application)} \\
&\equiv ((\lambda\, x\, y\, z.\, (x\, z)\, (y\, z))\, u)\, v\, w && \text{(left-associative application)} \\
&\equiv (((\lambda\, x\, y\, z.\, (x\, z)\, (y\, z))\, u)\, v)\, w && \text{(left-associative application)} \\
&\equiv (((\lambda\, x\, y\, z.\, ((x\, z)\, (y\, z)))\, u)\, v)\, w && \text{(application higher precedence)} \\
&\equiv (((\lambda\, x.\, \lambda\, y.\, \lambda\, z.\, ((x\, z)\, (y\, z)))\, u)\, v)\, w && \text{(right-associative } \lambda\text{-abstraction)} \\
&\equiv (((\lambda\, x.\, \lambda\, y.\, (\lambda\, z.\, ((x\, z)\, (y\, z))))\, u)\, v)\, w && \text{(right-associative } \lambda\text{-abstraction)} \\
&\equiv (((\lambda\, x.\, (\lambda\, y.\, (\lambda\, z.\, ((x\, z)\, (y\, z)))))\, u)\, v)\, w && \text{(right-associative } \lambda\text{-abstraction)} \\
&\equiv ((((\lambda\, x.\, (\lambda\, y.\, (\lambda\, z.\, ((x\, z)\, (y\, z)))))\, u)\, v)\, w) && \text{(remove outermost parentheses)}
\end{aligned}
$$

Binding

Definition

A variable x occurs **free** in M if x is not in the scope of λx. Otherwise, x occurs **bound**.

Definition

The **set of free variables in M**, denoted by $\mathrm{FV}(M)$, is inductively defined by

$$\begin{aligned}
\mathrm{FV}(x) &:= \{x\}, \\
\mathrm{FV}(M\,N) &:= \mathrm{FV}(M) \cup \mathrm{FV}(N), \\
\mathrm{FV}(\lambda x.M) &:= \mathrm{FV}(M) - \{x\}.
\end{aligned}$$

Substitution

Definition

The result of **substituting N for every free occurrence of x in M**, and changing bound variables to avoid clashes, denoted by $M[\,x \mapsto N\,]$, is defined by [Hindley and Seldin 2008, Definition 1.12]

$$
\begin{aligned}
x[\,x \mapsto N\,] &\coloneqq N; \\
y[\,x \mapsto N\,] &\coloneqq y, && \text{if } y \not\equiv x; \\
(P\,Q)[\,x \mapsto N\,] &\coloneqq P[\,x \mapsto N\,]\,Q[\,x \mapsto N\,]; \\
(\lambda x.P)[\,x \mapsto N\,] &\coloneqq \lambda x.P; \\
(\lambda y.P)[\,x \mapsto N\,] &\coloneqq \lambda y.P, && \text{if } y \not\equiv x \text{ and } x \notin \mathrm{FV}(P); \\
(\lambda y.P)[\,x \mapsto N\,] &\coloneqq \lambda y.P[\,x \mapsto N\,], && \text{if } y \not\equiv x, x \in \mathrm{FV}(P) \text{ and } y \notin \mathrm{FV}(N); \\
(\lambda y.P)[\,x \mapsto N\,] &\coloneqq \lambda z.P[\,x \mapsto N\,][\,y \mapsto z\,], && \text{if } y \not\equiv x, x \in \mathrm{FV}(P) \text{ and } y \in \mathrm{FV}(N);
\end{aligned}
$$

where in the last equation, the variable z is chosen such that $z \notin \mathrm{FV}(N\,P)$.

Substitution

Example

$(y\,(\lambda\, v.\, x\, v))[\, x \mapsto (\lambda\, y.\, v\, y)\,] \equiv y\,(\lambda\, z.\, (\lambda\, y.\, v\, y)\, z)$ (with $z \not\equiv v, y, x$).

Conversion Rules

Introduction

The functional behaviour of the λ-calculus is formalised through of their conversion rules:

$$\lambda x.N =_\alpha \lambda y.(N[\,x \mapsto y\,]) \qquad (\alpha\text{-conversion})$$

$$(\lambda x.M)\,N =_\beta M[\,x \mapsto N\,] \qquad (\beta\text{-conversion})$$

$$\lambda\,x.\,M\,x =_\eta M \qquad (\eta\text{-conversion})$$

Alpha Congruence

Definition

A **changed of bound variables** in M is to replace a subterm $\lambda x.N$ of M by $\lambda y.(N[x \mapsto y])$ where y does not occur in N.

Definition

A λ-term M is **α-congruent** with N, denoted by $M \equiv_\alpha N$, iff N results from M by a finite (perhaps empty) series of changes of bound variables.

Example

Whiteboard.

Alpha Congruence

Theorem

The relation $\equiv_\alpha$ is an equivalence relation.*

Convention

Following Barendregt [(1981) 2004, Convention 2.1.12], we syntactically identified λ-terms that are α-congruent, that is,

$$M \equiv N \coloneqq M \equiv_\alpha N.$$

*See, e.g. [Hindley and Seldin 2008, Lemma 1.19b].

Compatible Relations

Definition

A binary relation R on Λ is **compatible** iff*

$$(M, N) \in R \quad \Rightarrow \quad \begin{cases} (P\,M, P\,N) \in R, \\ (M\,P, N\,P) \in R, \\ (\lambda x.M, \lambda x.N) \in R. \end{cases}$$

*See, e.g. [Barendregt (1981) 2004, Definition 3.1.1i].

Beta Reduction

Definition

The binary relation β on Λ is defined by

$$\beta := \{\, ((\lambda x.M)\, N, M[\, x \mapsto N\,]) \mid M, N \in \Lambda \,\}.$$

Beta Reduction

Definition

The binary relation **one step β-reduction** on Λ, denoted by $\to_\beta$, is the compatible closure of β.

The $\to_\beta$ relation can be inductively defined by*

$$\frac{(M, N) \in \beta}{M \to_\beta N}$$

$$\frac{M \to_\beta N}{P\,M \to_\beta P\,N} \qquad \frac{M \to_\beta N}{M\,P \to_\beta N\,P} \qquad \frac{M \to_\beta N}{\lambda x.M \to_\beta \lambda x.N}$$

*See, e.g. [Barendregt (1981) 2004, Definition 3.1.5].

Beta Reduction

Definition

The binary relation **β-reduction** on Λ, denoted by $\twoheadrightarrow_\beta$, is the reflexive and transitive closure of $\to_\beta$.

The $\twoheadrightarrow_\beta$ relation can be inductively defined by*

$$\frac{M \to_\beta N}{M \twoheadrightarrow_\beta N}$$

$$\frac{}{M \twoheadrightarrow_\beta M} \qquad \frac{M \twoheadrightarrow_\beta N \qquad N \twoheadrightarrow_\beta P}{M \twoheadrightarrow_\beta P}$$

*See, e.g. [Barendregt (1981) 2004, Definition 3.1.5].

Beta Equality or Beta Convertibility

Definition

The binary relation β-**equality** (or β-**convertibility**) on Λ, denoted by $=_\beta$, is the equivalence relation generated by $\twoheadrightarrow_\beta$.

The $=_\beta$ relation can be inductively defined by*

$$\frac{M \twoheadrightarrow_\beta N}{M =_\beta N}$$

$$\frac{M =_\beta N}{N =_\beta M} \qquad \frac{M =_\beta N \qquad N =_\beta P}{M =_\beta P}$$

*See, e.g. [Barendregt (1981) 2004, Definition 3.1.5].

Normal Forms

Definition

A **$\boldsymbol{\beta}$-redex** is a λ-term of the form $(\lambda x.M)\,N$.

Definition

A λ-term which contains no β-redex is in **$\boldsymbol{\beta}$-normal form** (β-nf).

Definition

A λ-term N **is a $\boldsymbol{\beta}$-nf of** M (or M **has the $\boldsymbol{\beta}$-nf** M) iff N is a β-nf and $M =_\beta N$.

Example

Whiteboard.

Normal Forms

Remark

Church [1935, 1936] proved that the set

$$\{\, M \in \Lambda \mid M \text{ has a } \beta\text{-normal form} \,\}$$

is not computable* (i.e. undecidable). This was the first undecidable set ever.†

*We use the term ‘computable’ rather than ‘recursive’ following to [Soare 1996].

†See also [Barendregt 1990].

Combinators

Definition

A **combinator** (or **closed $\boldsymbol{\lambda}$-term**) is a λ-term without free variables.

Convention

A combinator called for example pred will be denoted by pred.

Combinators

Example

Some common combinators.

$\mathsf{B} := \lambda\, f\, g\, x.\, f\,(g\, x)$	(a composition combinator)
$\mathsf{B}' := \lambda\, f\, g\, x.\, g\,(f\, x)$	(a reversed composition combinator)
$\mathsf{C} := \lambda\, x\, y\, z.\, x\, z\, y$	(a permuting combinator)
$\mathsf{I} := \lambda x.x$	(an identity combinator)
$\mathsf{K} := \lambda\, x\, y.\, x$	(a projection combinator)
$\mathsf{M} := \lambda\, x.\, x\, x$	(a doubling combinator)
$\mathsf{S} := \lambda\, f\, g\, x.\, f\, x\,(g\, x)$	(a stronger composition combinator)
$\mathsf{T} := \lambda\, x\, y.\, y\, x$	(a permuting combinator)
$\mathsf{V} := \lambda\, x\, y\, z.\, z\, y\, x$	(a permuting combinator)
$\mathsf{W} := \lambda\, f\, x.\, f\, x\, x$	(a doubling combinator)

Combinators

Remark

The programs in a programming language based on λ-calculus are combinators.

Remark

The combinators K and S (i.e. the combinatory logic) are a Turing-complete language.

Fixed-Point Combinators

Definition

A **fixed-point combinator** is any combinator fix such that for all terms M,

$$\mathsf{fix}\, M =_\beta M\, (\mathsf{fix}\, M).$$

Theorem

The combinator $\mathsf{Y} := \lambda\, f.\, V\, V$, where $V \equiv \lambda\, x.\, f\, (x\, x)$, is a fixed-point combinator.*

Theorem

The combinator $\mathsf{U}\,\mathsf{U}$, where $\mathsf{U} := \lambda\, u\, x.\, x\, (u\, u\, x)$, is a fixed-point combinator.†

*According to [Hindley and Seldin 2008, p. 36], this combinator was hinted by Curry in 1929 and first published by Rosenbloom [1950]. See also [Barendregt (1981) 2004, Corollary 6.1.3].

†Defined by Turing [1937]. See, also [Barendregt (1981) 2004, Definition 6.1.4].

Recursion Using Fixed-Points

Example

An informal example using the factorial function [Peyton Jones 1987, § 2.4.1].

$$\begin{aligned}
\mathsf{fac} &\coloneqq \lambda\, n.\, \mathsf{if}\,(n == 0)\,\mathsf{then}\,1\,\mathsf{else}\,n * \mathsf{fac}\,(n-1) && \text{(combinator)} \\
&\equiv \lambda\, n.\, (\ldots \mathsf{fac} \ldots) && \text{(recursive combinator)} \\
&\equiv (\lambda\, f.\, \lambda\, n.\, (\ldots f \ldots))\, \mathsf{fac} && \text{(λ-abstraction on fac)}
\end{aligned}$$

Recursion Using Fixed-Points

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&\equiv \lambda\, n.\, (\ldots \mathsf{fac} \ldots) && \text{(recursive combinator)}\\
&\equiv (\lambda\, f.\, \lambda\, n.\, (\ldots f \ldots))\, \mathsf{fac} && \text{(λ-abstraction on fac)}
\end{aligned}
$$

Now, we can redefine the factorial function using fix.

$$\mathsf{h} := \lambda\, f.\, \lambda\, n.\, (\ldots f \ldots) \qquad \text{(non-recursive combinator)}$$

$$\mathsf{fac} := \mathsf{fix}\, \mathsf{h} \qquad \text{(fac is a fixed-point of h)}$$

(continued on next slide)

Recursion Using Fixed-Points

Example (continuation)

$$
\begin{aligned}
\mathsf{fac}\,1 &\equiv \mathsf{fix}\,\mathsf{h}\,1 \\
&=_\beta \mathsf{h}\,(\mathsf{fix}\,\mathsf{h})\,1 \\
&\equiv (\lambda\,f.\,\lambda\,n.\,(\ldots f \ldots))\,(\mathsf{fix}\,\mathsf{h})\,1 \\
&\twoheadrightarrow_\beta \mathsf{if}\,(1 == 0)\,\mathsf{then}\,1\,\mathsf{else}\,1 * (\mathsf{fix}\,\mathsf{h}\,0) \\
&\twoheadrightarrow_\beta 1 * (\mathsf{fix}\,\mathsf{h}\,0) \\
&=_\beta 1 * (\mathsf{h}(\mathsf{fix}\,\mathsf{h})\,0) \\
&\equiv 1 * ((\lambda\,f.\,\lambda\,n.\,(\ldots f \ldots))\,(\mathsf{fix}\,\mathsf{h})\,0) \\
&\twoheadrightarrow_\beta 1 * (\mathsf{if}\,(0 == 0)\,\mathsf{then}\,1\,\mathsf{else}\,1 * (\mathsf{fix}\,\mathsf{h}\,(-1))) \\
&\twoheadrightarrow_\beta 1 * 1 \\
&\twoheadrightarrow_\beta 1
\end{aligned}
$$

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