CM0884 Graph Theory

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Administrative Information

Course web page

http://www1.eafit.edu.co/asr/courses/cm0884-graph-theory/

Exams, bibliography, etc.

See course web page.

Textbook

Diestel, Reinhard (2017). Graph Theory. 5th ed. Springer.

The Basics

Preliminaries

Convention

The set of natural numbers, denoted $\mathbb{N}$, includes the zero.

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Definition

'A set $\mathcal{A} = \{A_1, \dots, A_k\}$ of disjoint subsets of a set A is a **partition** of A if the union $\bigcup A$ of all the sets $A_i \in A$ is A and $A_i \neq \emptyset$ for every i.' (Diestel 2017, p. 1)

Graphs

Definition

A **graph** (*grafo*) is an order pair $G = (V, E)$ of disjoint sets such that $E \subseteq [V]^2$.

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Definition

The **vertices** and the **edges** (*aristas*) of a graph $G = (V, E)$ are the elements of V and E, respectively.

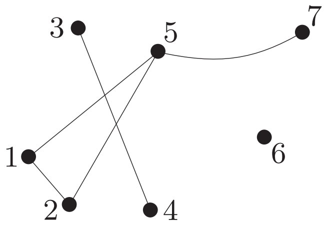
Notation

The vertex set of a graph G is denoted $V(G)$ and its edge set is denoted $E(G)$.

Graphs

Example

First example of textbook.*



$$V = \{1, \ldots, 7\},$$
$$E = \{\{1, 2\}, \{1, 5\}, \{2, 5\}, \{3, 4\}, \{5, 7\}\}.$$

*Figure source: Diestel (2017, Fig. 1.1.1).

Graphs

Example

- $G = (V, E)$ where $V = \{v_1, v_2, v_3\}$ and $E = [V]^2$.
- $G = (V, E)$ where $V = \{v_1, v_2, v_3\}$ and $E = \emptyset$.

Graphs

Definition

The **empty graph** is the graph $(\emptyset, \emptyset)$ and it is denoted $\emptyset$.

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Discussion

Should or should not the empty-graph be a graph?

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Discussion

Should or should not the empty-graph be a graph?

In the abstract of an article about this question, Harary and Read (1974) wrote:

> *'The graph with no points and no lines is discussed critically. Arguments for and against its official admittance as a graph are presented. This is accompanied by an extensive survey of the literature. Paradoxical properties of the null-graph are noted. No conclusion is reached.'*

Graphs

Some remarks on our definition of graph

- The edges in our graphs are undirected because $\{v, w\} = \{w, v\}$.

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- A graph with undirected egdes, without loops and without parallel edges is also called a **simple** graph in the literature. E.g. (Bondy and Murty 2008).

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- Since $\{v, v\} \notin [V]^2$ because $|\{v, v\}| = |\{v\}| = 1$, our graphs have no loops.
- Since the multiplicity of an element in a set is one, our graphs have no parallel edges.
- A graph with undirected egdes, without loops and without parallel edges is also called a **simple** graph in the literature. E.g. (Bondy and Murty 2008).
- The sets V and E must be disjoint for ruling out 'graphs' like $V = \{a, b, \{a, b\}\}$ and $E = \{\{a, b\}\}$.

Graphs

Definition

The **order** of graph G, denoted $|G|$, is its number of vertices.

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Definition

A graph G is **finite**, **infinite**, **countable** and so on according to $|G|$.

Graphs

Definition

A graph is **trivial** iff $|G| = 0$ or $|G| = 1$.

Remark

Diestel (2017, p. 16):

> *'Sometimes, e.g. to start an induction, trivial graphs can be useful; at other times they form silly counterexamples and become a nuisance. To avoid cluttering the text with non-triviality conditions, we shall mostly treat the trivial graphs, and particularly the empty graph $\emptyset$, with generous disregard.'*

Graphs

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Definition

The two vertices incident with an edge are its **ends** (*puntos finales*).

Graphs

Notation

An edge $\{x, y\}$ also will be written as xy (or yx).

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If $x \in X$ and $y \in Y$, then xy is an $\boldsymbol{X}$-$\boldsymbol{Y}$ **edge**.

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Definition

If $x \in X$ and $y \in Y$, then xy is an $\boldsymbol{X}$**-**$\boldsymbol{Y}$ **edge**.

Notation

The set of all X-Y edges in a set E is denoted by $E(X, Y)$. In addition.

$$E(x, Y) := E(\{x\}, Y),$$
$$E(X, y) := E(X, \{y\}).$$

Graphs

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Notation

The set of all the edges in a set E at a vertex v is denoted $E(v)$.

Graphs

Definition

Two vertices x, y of a graph G are **adjacent** or **neighbours**, iff $\{x, y\}$ is an edge of G.

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Two vertices x, y of a graph G are **adjacent** or **neighbours**, iff $\{x, y\}$ is an edge of G.

Definition

Two edges $e \neq f$ are **adjacent** iff they have an end in common.

Complete Graphs

Definition

A graph is **complete** iff all its vertices are pairwise adjacent.

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A graph is **complete** iff all its vertices are pairwise adjacent.

Notation

A complete graph on n vertices is denoted K^n.

Complete Graphs

Example (some K^n graphs*)

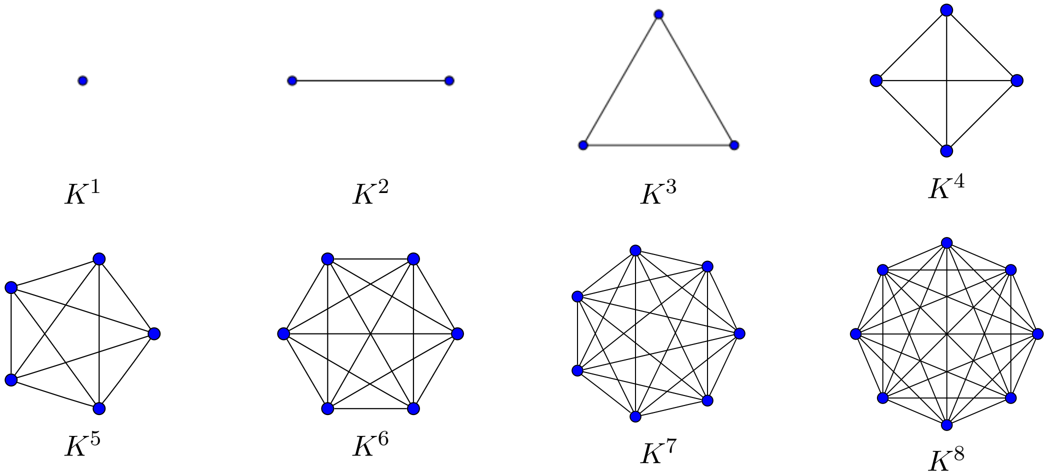


*Figure from: https://en.wikipedia.org/wiki/Complete_graph .

Isomorphisms

Definition

Whiteboard.

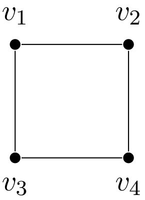
Notation

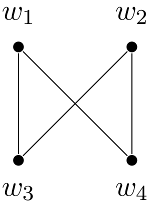
If G and G' are isomorphic is denoted by $G \simeq G'$.

Isomorphisms

Example

The following graphs are isomorphic.





The functions ψ and its inverse preserve adjacency.

$$\psi : \{v_1, v_2, v_3, v_4\} \to \{w_1, w_2, w_3, w_4\}$$

$\psi(v_1) = w_1$, $\psi(v_2) = w_4$, $\psi(v_3) = w_3$ and $\psi(v_4) = w_2$.

Isomorphisms

Remark

Since graphs are not algebraic structures but relational ones, a bijective homomorphism between graphs need not be an isomorphism (see, e.g. Cohn (1981, p. 190)).

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Definition

A homomorphism ψ between two relational structures is (see, e.g. Cohn (1981)):

- a **monomorphism** iff ψ is an injection,
- an **epimorphism** iff ψ is a surjection,
- an **endomorphism** iff ψ is from a relational structure to itself,
- an **isomorphism** iff ψ has an inverse which is also a homomorphism,
- an **automorphism** iff ψ is an isomorphism and an endomorphism.

Isomorphisms

Definition

A graph **property** is a class of graphs that is closed under isomorphism.

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Example

- To have a number even of vertices.
- To contain three pairwise adjacent vertices.

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Example

- The number of vertices (or edges).
- The greatest number of pairwise adjacent vertices.

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Notation

The expression $x := y$ means that x is being defined as y.

Isomorphisms

Definition

If A is a proposition then we define

$$\mathbf{1}(A) := \begin{cases} 1, & \text{if } A \text{ is true;} \\ 0, & \text{otherwise.} \end{cases}$$

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The **indicator** function of a set S, denoted $\mathbf{1}_S$, is defined by (Lovász 2012):

$$\mathbf{1}_S : S \to \{0, 1\}$$
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Relation between properties and invariants

If we identify a graph property P with its indicator function $\mathbf{1}_P$, graph properties are just 0-1 valued graph invariants (Lovász 2012).

Subgraphs

Definition

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs.

If $V' \subseteq V$ and $E' \subseteq E$ then G' is a **subgraph** of G, denoted by $G' \subseteq G$, and G is a **supergraph** of G'.

If $G' \subseteq G$ and $G' \neq G$, then G' is a **proper** subgraph of G.

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Examples

Whiteboard.

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If $G' \subseteq G$ and $G' \neq G$, then G' is a **proper** subgraph of G.

Examples

Whiteboard.

Theorem

The subgraph relation forms a partial order on all graphs.

Proof

Whiteboard.

Subgraphs

Definition

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. If $G' \subseteq G$ and G' contains all the edges whose ends are both in V', then G' is an **induced subgraph** of G and V' **induces** or **spans** G' in G.

Remark

Note that an induced subgraph is a subgraph obtained only by deleting vertices.

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Examples

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Notation

Let $G = (V, E)$ be a graph and $U \subseteq V$ a set of vertices. We write $G[U]$ for the graph on U whose edges are the edges of G with both ends in U.

Subgraphs

Definition

If $G' \subseteq G$ and G' contains all the vertices of G, then G' is a **spanning subgraph** (*subgrafo de expansión*) of G.

Remark

Note that a spanning subgraph is a subgraph obtained only by deleting edges.

Remark

Note that every graph is a spanning subgraph of a complete graph.

New Graphs Using Operations between Sets

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs.

Definition

We define the following graphs:

$$G \cup G' := (V \cup V', E \cup E'),$$
$$G \cap G' := (V \cap V', E \cap E'),$$
$$\overline{G} := (V, [V]^2 \setminus E).$$

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\end{aligned}
$$

Definition

If $G \cap G' = \emptyset$, then G and G' are **disjoint** graphs.

New Graphs Using Operations between Sets

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs, $U \subseteq V$ a set of vertices and $F \subseteq [V]^2$ a set of edges.

Definition

We define the following graphs:

$$G - U := G[V \setminus U], \qquad G - v := G - \{v\},$$
$$G - G' := G - V(G'),$$

$$G - F := (V, E \setminus F), \qquad G - e := G - \{e\},$$
$$G + F := (V, E \cup F), \qquad G + e := G + \{e\}.$$

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Examples

Whiteboard.

New Graphs Using Other Operations

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Let G and G' be two disjoint graphs. By $G * G'$ we denoted the graph obtained from $G \cup G'$ by joining all the vertices of G to all the vertices of G'.

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Example

$K^2 * K^3 = K^5$.

The Degree of a Vertex

Definition

The **degree** (or **valence**) of a vertex v, denoted $d(v)$, is the number of edges at v.

The Degree of a Vertex

Definition

We define the following invariants:

$$\delta(G) := \min \{d(v) \mid v \in V\} \qquad \text{(\textbf{minimum degree} of } G\text{)}$$

$$\Delta(G) := \max \{d(v) \mid v \in V\} \qquad \text{(\textbf{maximum degree} of } G\text{)}$$

$$d(G) := \frac{1}{|V|} \sum_{v \in V} d(v) \qquad \text{(\textbf{average degree} of } G\text{)}$$

$$\epsilon(G) := \frac{|E|}{|V|} \qquad \text{(\textbf{number of edges by vertex} in } G\text{)}$$

The Degree of a Vertex

Proposition (1.2.2)

Every graph G with at least one edge has a subgraph H with

$$\delta(H) > \epsilon(H) \geq \epsilon(G).$$

Proof

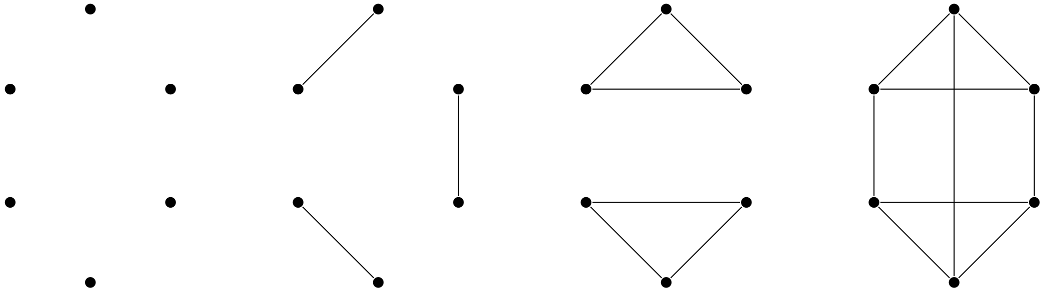
Whiteboard.

Regular Graphs

Definition

For $k \in \mathbb{N}$, a graph is ***k*-regular** iff all its vertices have degree k.

Example (some regular graphs*)



0-regular 1-regular 2-regular 3-regular

*Example from https://en.wikipedia.org/wiki/Regular_graph .

Regular Graphs

Convention

A 3-regular graph is called a **cubic** graph.

Example

The Petersen graph and the Heawood graph are cubic graphs.

Paths and Cycles

Definition

A **path** is a non-empty graph $P = (V, E)$ where V and E are of the form

$$\begin{aligned} V &= \{x_0, x_1, \ldots, x_k\}, \\ E &= \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}, \end{aligned}$$

and the x_i are all distinct.

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and the x_i are all distinct. The vertices x_0 and x_k are the **ends** of P.

Paths and Cycles

Definition

The **length** of a path is its number of edges.

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Example

$P^0 = K^1$.

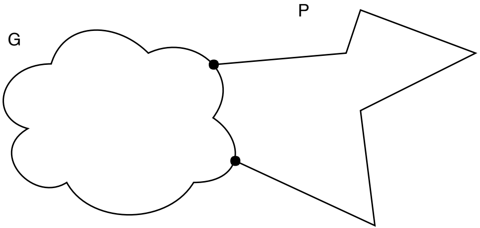
Examples

Whiteboard.

Paths and Cycles

Definition

Let G be a graph. A path P is a $\boldsymbol{G}$**-path** iff P is non-trivial and meets G exactly in its ends.



Paths and Cycles

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Notation

Let $P = x_0x_1 \dots x_k$ be a path. For $0 \leq i \leq j \leq k$ we write

$$x_iPx_j := x_i \dots x_j.$$

Paths and Cycles

Definition

Let A and B two sets of vertices. A path $P = x_0 \ldots x_k$ is an $\boldsymbol{A}$-$\boldsymbol{B}$ **path** if

$$V(P) \cap A = \{x_0\} \quad \text{and} \quad V(P) \cap B = \{x_k\}.$$

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Notation

$$\begin{aligned} a\text{-}B \text{ path } &:= \{a\}\text{-}B \text{ path}, \\ A\text{-}b \text{ path } &:= A\text{-}\{b\} \text{ path}, \\ a\text{-}b \text{ path } &:= \{a\}\text{-}\{b\} \text{ path}. \end{aligned}$$

Paths and Cycles

Definition

If $P = x_0 \ldots x_{k-1}$ is a path and $k \geq 3$, then the following graph is a **cycle**:

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The **length** of a cycle is its number of edges (or vertices).

Paths and Cycles

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If $P = x_0 \dots x_{k-1}$ is a path and $k \geq 3$, then the following graph is a **cycle**:

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Definition

The **length** of a cycle is its number of edges (or vertices). Let $k \geq 3$ be an integer. A cycle of length k is a ***k*-cycle** and it is denoted by C^k.

Examples

Whiteboard.

Paths and Cycles

Definition

An **acyclic** graph is a graph that contains no cycles.

Paths and Cycles

Definition

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Definition

An **even/odd cycle** is a cycle of even/odd length.

Paths and Cycles

Definition

The **girth** (*cintura*) of a graph G, denoted $g(G)$, is the length of a shortest cycle contained in G. If G does not contain a cycle, then $g(G) := \infty$.

Remark

Note that we wrote 'a shortest cycle' instead of 'the shortest cycle'.

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Example

The girths of the Petersen graph and the Heawood graph are 5 and 6, respectively.

Paths and Cycles

Definition

Let C be a cycle. A **chord** of C is an edge which connects two vertices of C but it is not part of C.

Paths and Cycles

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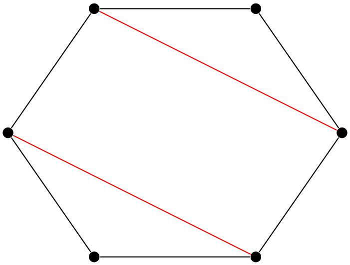
Example

A cycle (black) with two chords (red edges).



Paths and Cycles

Definition

Let G be a graph. An **induced cycle** in G is a cycle in G that has no chords.

Remark

Note that an induce cycle in G is an induced subgraph in G.

Example

See Diestel (2017, Fig. 1.3.3).

Paths and Cycles

Proposition (1.3.1)

Let G be a graph. If $\delta(G) \geq 2$ then G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$.

Paths and Cycles

Definition

The **distance** between two vertices x, y, denoted $d(x, y)$, is the length of a shortest x-y path in a graph. If there is no such path, then $d(x, y) := \infty$.

Remark

Note that we wrote ‘a shortest x-y path’ instead of ‘the shortest x-y path’.

Paths and Cycles

Definition

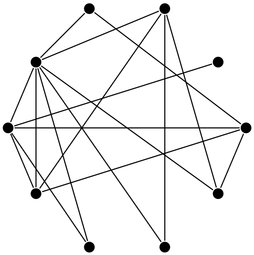
Let G be a graph. The **diameter** of G, denoted $\operatorname{diam}(G)$, is the greatest distance between any two vertices in G, that is,

$$\operatorname{diam}(G) := \max_{x,y \in V(G)} d(x,y).$$

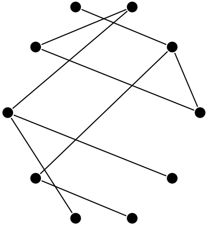
Paths and Cycles

Example

Graphs on 10 vertices with diameters 3 and 7.*



Diameter 3



Diameter 7

*Example from http://mathworld.wolfram.com/GraphDiameter.html .

Paths and Cycles

Proposition (1.3.2)

Every graph G containing a cycle satisfies

$$g(G) \leq 2\,\mathrm{diam}(G) + 1.$$

Proof by contradiction

Whiteboard.

Paths and Cycles

Definition

Let G be a graph. The **eccentricity** of a vertex x, denoted $\mathrm{ecc}(x)$, is the greatest distance between x and any other vertex of G (see, e.g. Harary (1969)), that is,

$$\mathrm{ecc}(x) := \max\,\{d(x, y) \mid y \in V(G)\}.$$

Paths and Cycles

Example

A graph where the eccentricity of each vertex is shown.*

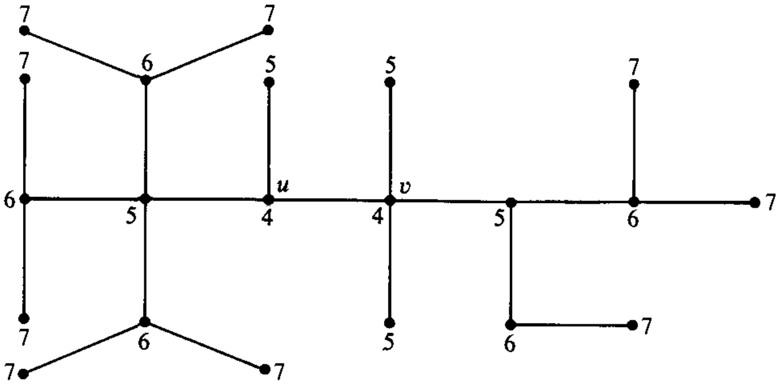


*Figure source: Harary (1969, Fig. 4.2).

Paths and Cycles

Definition

Let G be a graph. The **radius** of G, denoted $\mathrm{rad}(G)$, is the minimum eccentricity of its vertices, that is,

$$\begin{aligned}\mathrm{rad}(G) &:= \min_{x \in V(G)} \max_{y \in V(G)} d(x, y) \\ &= \min \{\mathrm{ecc}(x) \mid x \in V(G)\}.\end{aligned}$$

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Definition

Let G be a graph. A vertex v is **central** in G iff $\mathrm{ecc}(v) = \mathrm{rad}(G)$.

Paths and Cycles

Example

A graph where the eccentricity of each vertex is shown, with $\mathrm{rad}(G) = 4$ and central vertices u and v.*

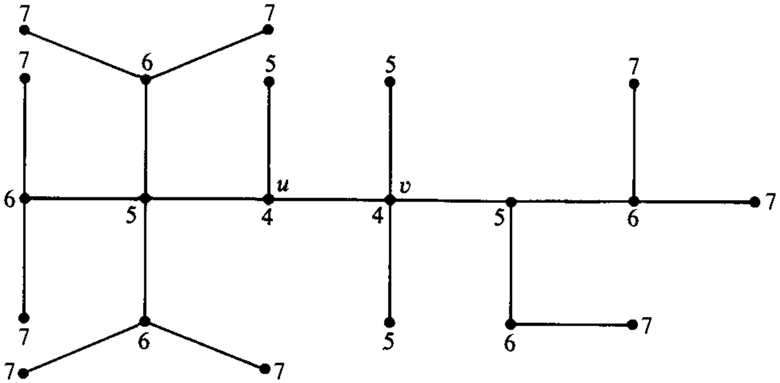

*Figure source: Harary (1969, Fig. 4.2).

Paths and Cycles

Definition

Let G be a graph. A **walk** of length k in G is a non-empty alternating sequence $v_0e_0v_1e_1 \ldots e_{k-1}v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$, for all $i < k$.

Connected Graphs

Definition

A non-empty graph G is **connected** (*conexo*) iff any two of its vertices are linked by a path in G. If a graph is not connected it is a **disconnected** graph.

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Complete graphs are connected graphs. The trivial non-empty graph is a connected graph.

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The non-trivial 0-regulars graphs are disconnected graphs.

Example

Let n be a positive integer. An n-regular graph can be a connected or a disconnected graph.

Connected Graphs

Example

The Petersen graph and the Heawood graph are connected graphs.

Connected Graphs

Proposition (1.4.1)

The vertices of a connected graph G can always be enumerated, say as $v_1, \ldots, v_n$, so that $G_i := G[v_1, \ldots, v_i]$ is connected for every i.

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Proof by induction on $|G|$.

1. Choose any vertex as v_1, so $G_1 := G[v_1]$ is connected.
2. Inductive hypothesis: Assume that $v_1, \dots, v_i$ have been chosen for some $i < |G|$, and $G_i := G[v_1, \dots, v_i]$ are connected.
3. Now, chose a vertex $v \in G - G_i$.
4. Since G is connected it contains a v-v_1 path P.
5. Choose as v_{i+1} the last vertex of P in $G - G_i$.
6. Then v_{i+1} has a neighbour in G_i.
7. The connectedness of G_{i+1} follows by the inductive hypothesis and the previous steps. ∎

Components

Remark

Recall that the subgraph relation forms a partial order on all graphs (see this theorem).

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The components of a graph are induced subgraphs.

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Let $G = (V, E)$ be a graph. The vertex sets of the components of G partition the set V.

Components

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Let $G = (V, E)$ be a graph. The vertex sets of the components of G partition the set V.

What about the empty-graph?

If a graph is connected then it is a not-empty graph, so the empty-graph has no components.

k-Connected Graphs

Definition

For $k \in \mathbb{N}$, a graph $G = (V, E)$ is **k-connected** (or **k-vertex-connected**) iff

i) $|G| > k$ and

ii) for every $X \subseteq V(G)$, if $|X| < k$ then $G - X$ is a connected graph.

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Every non-empty disconneted graph is 0-connected (by false antecedent).x

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The 1-connected graphs are the non-trivial connected graphs.

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A K^n graph, with $n \geq 1$, is an $(n-1)$-connected graph.

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To build a 1-connected but not 2-connected graph.

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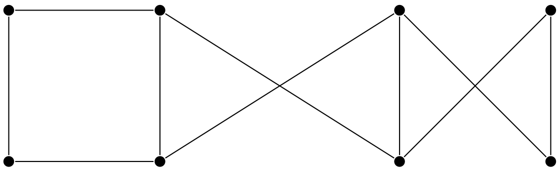
To build a 1-connected but not 2-connected graph.

Example

A 2-connected but not 3-connected graph.



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Why $\kappa(K^1)$ is not 1, but 0?

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Example

$\kappa(K^n) = n - 1$, for all $n \geq 1$.

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For $l \in \mathbb{N}$, a graph $G = (V, E)$ is **l-edge-connected** iff

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$\lambda(K^n) = n - 1$, for all $n \geq 2$.

Connectivity and Edge-Connectivity

Question

Can the connectivity and the edge-connectivity be equals? Yes!

*Figure source: Diestel (2017, Fig. 1.4.3).

Connectivity and Edge-Connectivity

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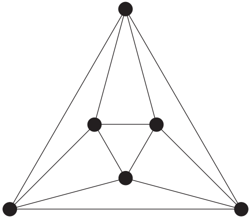
Can the connectivity and the edge-connectivity be equals? Yes!

Example

A graph G with $\kappa(G) = \lambda(G) = 4$.*



G

*Figure source: Diestel (2017, Fig. 1.4.3).

Connectivity and Edge-Connectivity

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Can the connectivity be smaller than the edge-connectivity? Yes!

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Connectivity and Edge-Connectivity

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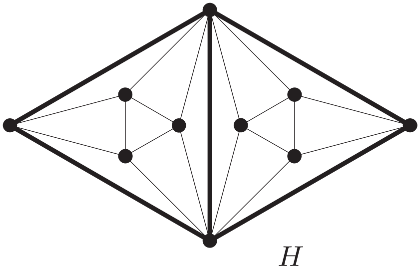
Can the connectivity be smaller than the edge-connectivity? Yes!

Example

A graph H with $\kappa(H) = 2$ and $\lambda(H) = 4$.*



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Edge-Connectivity and Minimum Degree

Question

Can the edge-connectivity and the minimum degree be equals? Yes!

Edge-Connectivity and Minimum Degree

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Can the edge-connectivity and the minimum degree be equals? Yes!

Remark

Recall that $\delta(G)$ denotes the minimum degree of a graph G.

Example

Let G be a K^n graph, with $n \geq 2$, then $\lambda(G) = \delta(G) = n - 1$.

Edge-Connectivity and Minimum Degree

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Can the edge-connectivity be smaller that the minimum degree? Yes!

Edge-Connectivity and Minimum Degree

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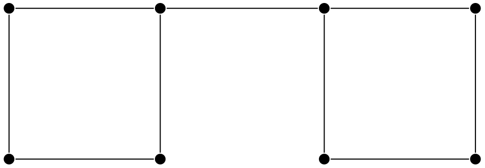
Can the edge-connectivity be smaller that the minimum degree? Yes!

Example

A graph G with $\lambda(G) = 1$ and $\delta(G) = 2$.



Connectivity, Edge-Connectivity and Minimum Degree

Proposition (1.4.2)

If G is non-trivial, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Forests and Trees

Definition

A **forest** (*bosque*) is an acyclic graph. A **tree** is a connected acyclic graph.

Forests and Trees

Definition

A **forest** (*bosque*) is an acyclic graph. A **tree** is a connected acyclic graph.

Remark

A forest is a graph whose components are trees.

Forests and Trees

Example

A forest formed with all the trees with at most five vertices.*

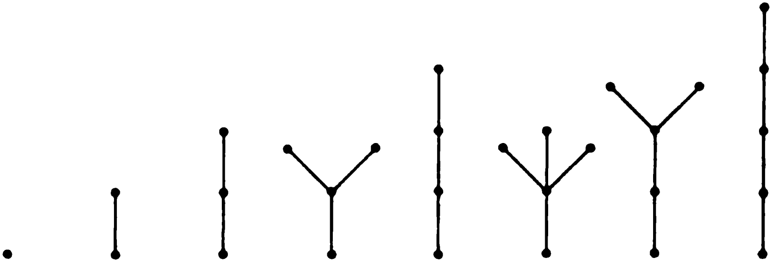


*Figure source: Biggs, Lloyd and Wilson (1998, Fig. 3.1).

Forests and Trees

Example

A tree.*

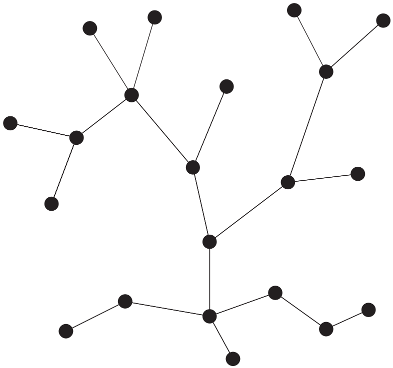


*Figure source: Diestel (2017, Fig. 1.5.1).

Forests and Trees

Definition

In a tree, the vertices of degree 1 are its **leaves** $($except when the root of tree exists and it has degree $1)$, the other vertices are its **inner vertices**.

Forests and Trees

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In a tree, the vertices of degree 1 are its **leaves** (except when the root of tree exists and it has degree 1), the other vertices are its **inner vertices**.

Remark

Every non-trivial tree has a leaf, so if we remove a leaf from a tree, we still have a tree.

Forests and Trees

Theorem (1.5.1)

The following four assertions are equivalent for a graph T:

i) T is a tree;

Forests and Trees

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Forests and Trees

Theorem (1.5.1)

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i) T is a tree;

ii) any two vertices of T are linked by a unique path in T;

iii) T is minimally (respect to the subgraph relation) connected, i.e. T is connected but $T - e$ is disconnected for every edge $e \in T$;

iv) T is maximally (respect to the subgraph relation) acyclic, i.e. T contains no cycle but $T + xy$ does, for any two non-adjacent vertices $x, y \in T$.

Forests and Trees

Corollary (1.5.2)

The vertices of a tree can always be enumerated, say as $v_1, \ldots, v_n$, so that every v_i with $i \geq 2$ has a unique neighbour in $\{v_1, \ldots, v_{i-1}\}$.

Proof

Use Proposition 1.4.1.

Spanning Trees

Definition

A **spanning tree** (*árbol de expansión, árbol generador o árbol recubridor*) of a graph G is a spanning subgraph of G which is a tree.

Spanning Trees

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Example

See `https://www.cs.usfca.edu/~galles/visualization/DFS.html` .

Spanning Trees

Example

Spanning trees for the graphs with red vertices.*

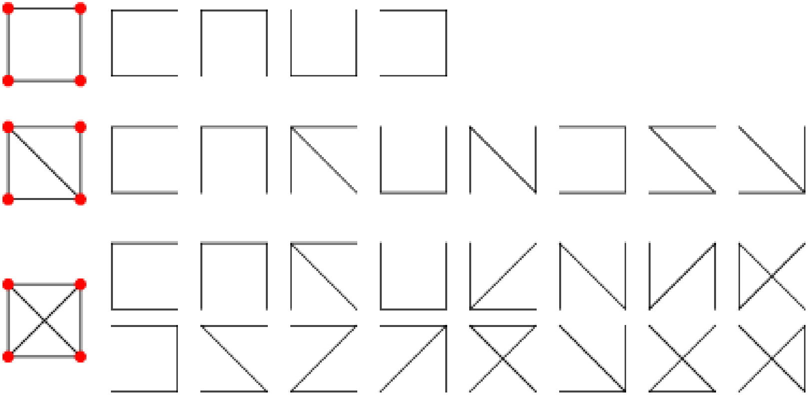


*Figure from http://mathworld.wolfram.com/SpanningTree.html .

Spanning Trees

Corollary

Every connected graph contains a spanning tree.

Spanning Trees

Corollary

Every connected graph contains a spanning tree.

Proof

- Build a minimal connected subgraph and apply Theorem 1.5.1.iii or
- Build a maximal acyclic subgraph and apply Theorem 1.5.1.iv.

Rooted Trees

Definition

The **root** of a tree is a vertex considered as special.

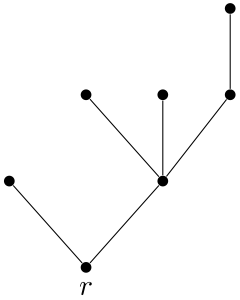
Definition

A **rooted tree** is a tree with a fixed root r.

Rooted Trees

Example

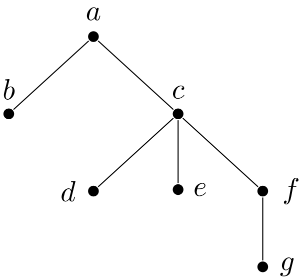
A tree with root r.



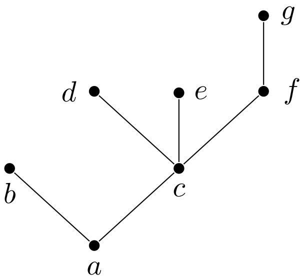
Rooted Trees

Example

Four representations of a tree with root a.*



Downward



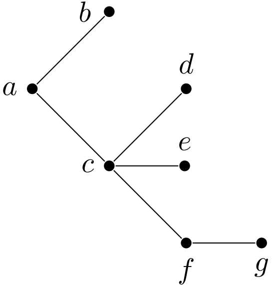
Upward

Continued on next slide

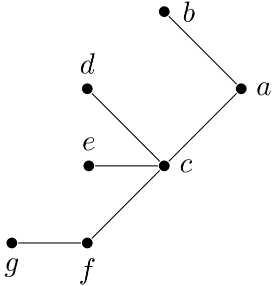
*Based on the discussion about how to draw trees in (Knuth 1997, § 2.3).

Rooted Trees

Example (continuation)



Left to right



Right to left

Rooted Trees

Definition

Let T be a tree with root r. We define the **tree-order** on $V(T)$ associated with T and r by:

$$x \leq y := x \in rTy.$$

Rooted Trees

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Theorem

The tree-order associated with a rooted tree T is a partial order on $V(T)$.

Proof

Whiteboard.

Rooted Trees

Remarks

In the tree-order of a rooted tree:

- the root is the least element,
- the leaves are its maximal elements and
- the ends of any edge are comparable

Rooted Trees

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The **down-closure** of a vertex v is defined by

$$\lceil v \rceil := \{x \mid x \leq v\}.$$

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The **down-closure** of a vertex v is defined by

$$\lceil v \rceil := \{x \mid x \leq v\}.$$

Remark

Let v be a vertex. The down-closure of v is a chain.

Normal Spanning Trees

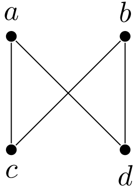
Definition

A rooted spanning tree T of a graph G is a **normal spanning tree** (also called **Trémaux tree**) iff the ends of every edge of G (i.e. every two adjacent vertices) are comparable in the tree-order on $V(G)$ induced by T (Diestel and Leader 2001).

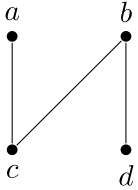
Normal Spanning Trees

Example

A graph G and a spanning tree T of G. The tree T is a normal spanning tree only when its root is a or d.



G



T

Normal Spanning Trees

Example

A normal spanning tree T with root r of the graph G.*

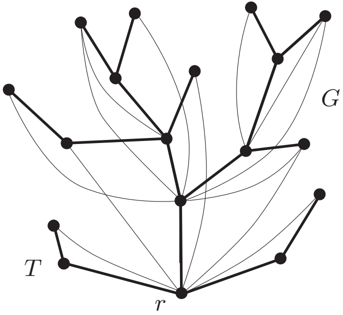


*Figure source: Diestel (2017, Fig. 1.5.2).

Normal Spanning Trees

Proposition (1.5.6)

Every connected graph contains a normal spanning tree, with any specified vertex as its root.

Normal Spanning Trees

Exercise (1.26^{+})

Depth-first search algorithm: Let G be a connected graph, and let $r \in G$ be a vertex. Starting from r, move along the edges of G, going whenever possible to a vertex not visited so far. If there is no such vertex, go back along the edge by which the current vertex was first reached (unless the current vertex is r; then stop).

Show that the edges traversed form a normal spanning tree in G with root r.

Normal Spanning Trees

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Show that the edges traversed form a normal spanning tree in G with root r.

Remark

See an animation of the depth-first search algorithm in https://www.cs.usfca.edu/~galles/visualization/DFS.html .

Normal Spanning Trees

Remark

The property of being a normal spanning tree can be expressed in monadic second-order logic (Courcelle and Engelfriet 2012).

Normal Spanning Trees

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Remark

A formal verification of deep-first search algorithms was done by Lammich and Neumann (2015).

Bipartite Graphs

Definition

Let $r \geq 2$ be an integer. A graph is **r-partite** iff its vertex set can be partitioned into r classes such that every edge has its ends in different classes.

Remark

Note that in an r-partite graph the vertices in the same partition class are not adjacent.

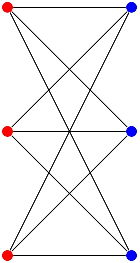
Convention

The 2-partite and 3-partite graphs are called **bipartite** and **tripartite**, respectively.

Bipartite Graphs

Example

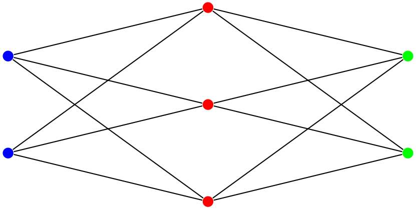
A bipartite graph.



Bipartite Graphs

Example

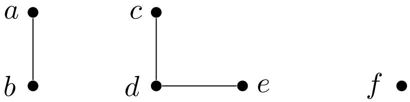
A tripartite graph.



Bipartite Graphs

Example

Is the following graph, which has three components, a bipartite graph?



Bipartite Graphs

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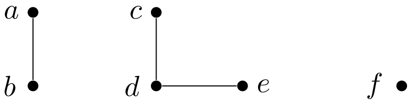
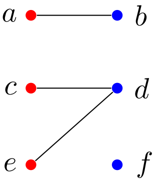


Yes!



Bipartite Graphs

Example

Another bipartite graph.

Bipartite Graphs

Proposition (1.6.1)

A graph is bipartite iff all its cycles are of even length.

Bipartite Graphs

Proposition (1.6.1)

A graph is bipartite iff all its cycles are of even length.

Proof ($\Rightarrow$).

See (Bollobás 2002, Theorem I.4).

1. Let G be a bipartite graph with two vertex classes V_1 and V_2 and let $C := x_1x_2\dots x_kx_1$ be a cycle in G.
2. We suppose that $x_1 \in V_1$ (if not, just rename the vertex classes).
3. Therefore, $x_2 \in V_2$, $x_3 \in V_1$, and so on. Hence, $x_i \in V_1$ iff i is odd.
4. Since $x_k \in V_2$, we can conclude that C is an even cycle. ■

Bipartite Graphs

Proof ($\Leftarrow$).

1. Let every cycle of G an even cycle and let suppose G is connected.
2. Pick a vertex $x \in V(G)$.
3. Define the vertex sets

$$V_1 := \{y \in V(G) \mid d(x, y) \text{ is odd}\},$$
$$V_2 := V(G) \setminus V_1.$$

4. Note that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V(G)$ because G is connected. Therefore, the set $\{V_1, V_2\}$ partitions $V(G)$.
5. Note that if G had any edge between two vertices of the same set V, the graph G would have an odd cycle.
6. Hence, G is bipartite.
7. Now, if G is disconnected build the partition repeating of previous steps on each component of G.

Bipartite Graphs

Proposition (1.6.1, previous version)

A graph is bipartite iff all its cycles are of even length.

Bipartite Graphs

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A graph is bipartite iff all its cycles are of even length.

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Discussion

Which is the smallest bipartite graph?

Bipartite Graphs

Definition

An r-partite graph is **complete** iff every two vertices from different partition classes are adjacent.

Definition

A **complete multipartite graph** is a graph that is complete k-partite for some k.

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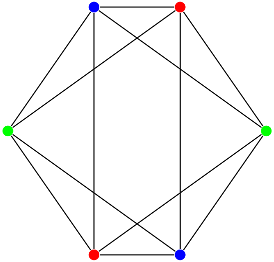
Convention

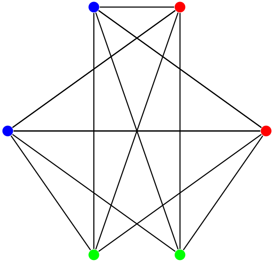
We denoted by $K_{n_1,\ldots,n_r}$ the complete r-partite graph where $n_1, \ldots, n_r$ are the sizes of each vertex set in the partition.

Bipartite Graphs

Example

Two drawings of the tripartite graph $K_{2,2,2}$.

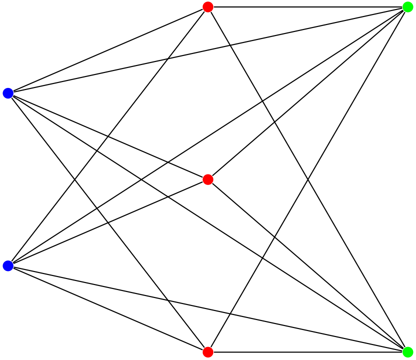




Bipartite Graphs

Example

The tripartite graph $K_{2,3,2}$.



Bipartite Graphs

Notation

The complete r-partite graph $\overline{K^{n_1}} * \cdots * \overline{K^{n_r}}$ is denoted by $K_{n_1,\ldots,n_r}$.

Representating Graphs

Let G be a graph with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$.

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Definition

The **incidence matrix** $B = (b_{ij})_{n \times m}$ of G is defined by

$$b_{ij} := \begin{cases} 1, & \text{if } v_i \in e_j; \\ 0, & \text{otherwise.} \end{cases}$$

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Definition

The **adjacency matrix** $A = (a_{ij})_{n\times n}$ of G is defined by

$$a_{ij} := \begin{cases} 1, & \text{if } v_i v_j \in E; \\ 0, & \text{otherwise.} \end{cases}$$

Other Notions of Graphs

Notation

Let A be a set. The power set of A is denoted by $\mathcal{P}(A)$.

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A **hypergraph** is an order pair (V, E) of disjoint sets of **vertices** and **edges** such that the elements of E are non-empty subsets of V, i.e. $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$.

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Remark

Note that graphs are hypergraphs where the elements of their edge sets have cardinality two.

Other Notions of Graphs

Example

Hypergraph example.*

$V = \{1, \dots, 7\}$, $E = \{e_1, \dots, e_6\}$, and

$$\begin{aligned} e_1 &= \{4, 5, 6\}, \\ e_2 &= \{1, 2\}, \\ e_3 &= \{1, 5, 6\}, \\ e_4 &= \{2, 3, 4\}, \\ e_5 &= \{4, 7\}, \\ e_6 &= \{3\}. \end{aligned}$$

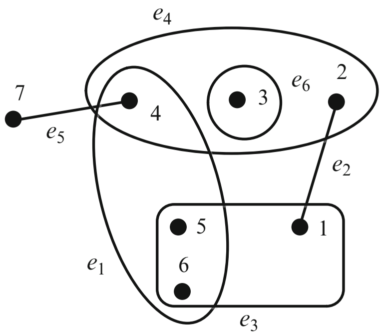

*Figure source: H. Zhang et al. (2018, Fig. 1.1).

Other Notions of Graphs

Definition

A **directed graph** (or **digraph**) is a order pair (V, E) of disjoint sets of **vertices** and **edges** and two functions $\mathrm{init} : E \to V$ and $\mathrm{ter} : E \to V$.

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Two or more edges are **multiple edges** iff they are edges between the same pair of vertices. If also they have the same direction they are **parallel edges**.

An edge e is a **loop** iff $\mathrm{init}(e) = \mathrm{ter}(e)$.

Other Notions of Graphs

Definition

A **multigraph** is a order pair (V, E) of disjoint sets of **vertices** and **edges** and one function $E \to V \cup [V]^2$.

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Remark

Note that multigraphs can have loops and multiple edges.

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Remark

Note that multigraphs can have loops and multiple edges.

Remark

Note that graphs are multigraphs without loops or multiple edges.

Colouring

Colouring Graphs

Definition

Let $G = (V, E)$ be a graph and S be a set whose elements are the available **colours**. A **vertex colouring** of G is a function

$$c : V \to S$$

such that $c(v) \neq c(w)$ whenever v and w are adjacent.

Colouring Graphs

Definition

A $\boldsymbol{k}$**-colouring** of a graph $G = (V, E)$ is a vertex colouring

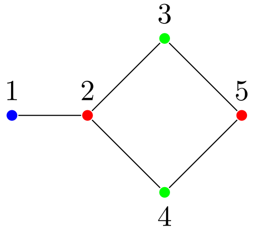
$$c : V \to \{1, \ldots, k\}$$

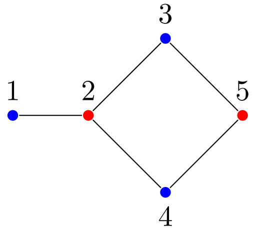
of G.

Colouring Graphs

Example

A 3-colouring and a 2-colouring for the same graph.





Colouring Graphs

Definition

Let G be a graph and k be the smallest integer such that G has a k-colouring. The number k is the **(vertex-)chromatic number** of G; denoted $\chi(G)$.

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Definition

A graph G is ***k*-colourable** iff $\chi(G) \leq k$.

Colouring Planar Graphs

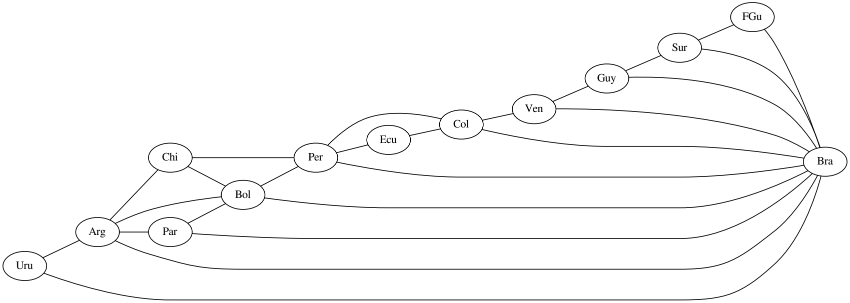
Example

How many colours do we need for colouring South America's map?

Colouring Planar Graphs

Example (continuation)

Graph associated with South America's map where the vertices are the countries and one edge between two vertices means that the countries share a border.

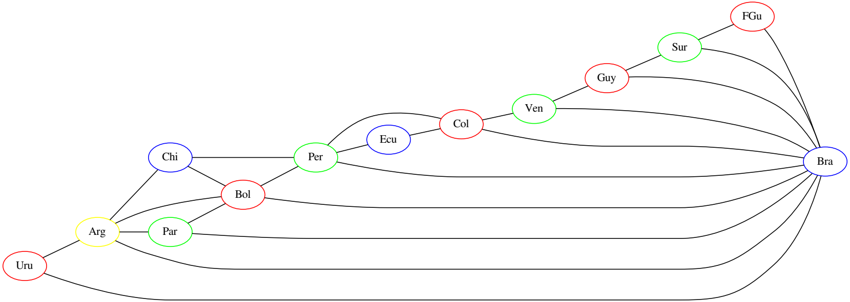


Continued on next slide

Colouring Planar Graphs

Example (continuation)

Let G be the associated graph with South America's map. Then $\chi(G) = 4$.



Colouring Planar Graphs

Theorem (Four colour theorem, 5.1.1)

Every planar graph is 4-colourable.

Flows

Flows in Networks

Theorem (Max-flow min-cut theorem, 6.2.2)

In a network, the maximum total value of a flow equals the minimum capacity of a cut (Ford and Fulkerson 1956).

Infinite Graphs

Example

The square, triangular and hexagonal lattices.*

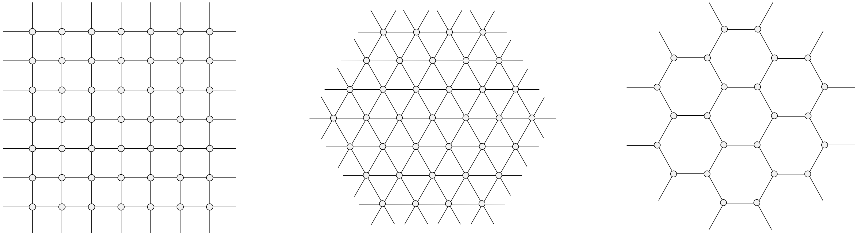


*Figure source: Bondy and Murty (2008, Fig. 1.27).

Ramsey Theory for Graphs

Ramsey's Original Theorems

Theorem (9.1.1)

For every $r \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that every graph of order at least n contains either K^r or $\overline{K^r}$ as an induced subgraph. By Ramsey (1930).

Historical Remarks

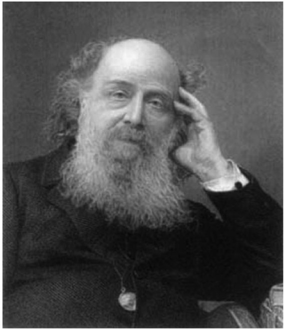
Historical Remarks

On the term ‘graph’

Sylvester (1878) introduced the term ‘graph’ on a note in Nature in the context of mathematics and chemistry (Biggs, Lloyd and Wilson 1998).

In relation to this term, the above authors wrote (p. 65):

‘So the credit (or blame) for the use of this term must be ascribed to Sylvester.’



James Joseph Sylvester (1814–1897)*

*Image from the MacTutor History of Mathematics Archive.

Historical Remarks

Previous definitions of graphs

König (1916, p. 453):

> *'Es sei eine endliche Anzahl yon Punkten gegeben; gewisse Paare, die man aus diesen Punkten auswiihlen kann, sollen durch eine oder mehrere (endlich viele) Kanten verbunden werden. Eine auf diese Weise entstehende Figur wird im allgemeinen als ein Graph bezeichne.'*

Translation (Biggs, Lloyd and Wilson 1998, p. 203):

> *'Let a finite number of points be given: then one can choose certain pairs of the points so that one or more (but finitely many) edges join them. A figure constructed in this way we shall generally call a graph.'*

Historical Remarks

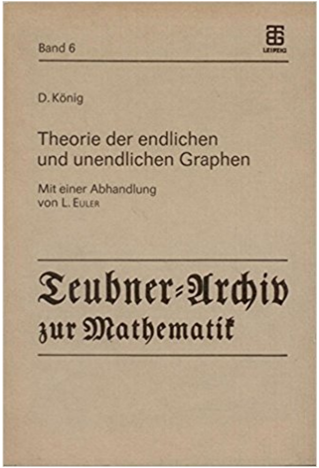
Previous definitions of graphs

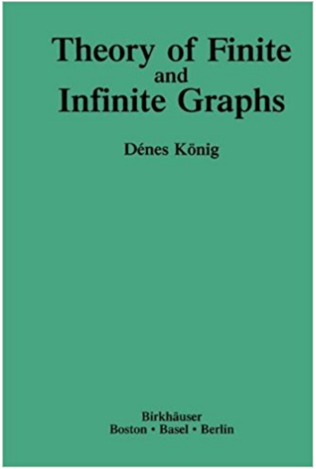
Whitney (1931, p. 378):

> *'Let a finite number of curves, or edges, whose end-points we call vertices, intersect at no other points than these vertices. Let the system be connected, that is, any two vertices are joined by a succession of edges, each two successive edges having a vertex in common. This forms a graph.'*

Historical Remarks

The first book on graph theory was written in German by König in 1936 and only translated to English in 1990





Logic of Graphs

Logic of Graphs

First-order logics

In the first-order theory of graphs, the universe of discourse are the vertices and the language has a proper binary predicate edg representing the relation of adjacency between vertices.

Example

Our definition of graph satisfies the following axioms (Goldberg 1993):

$$\begin{aligned} \forall v(\neg \mathrm{edg}(v,v)) \quad & \text{(no loops)} \\ \forall v \forall w(\mathrm{edg}(v,w) \Rightarrow \mathrm{edg}(w,v)) \quad & \text{(edges are undirected)} \end{aligned}$$

Logic of Graphs

Monadic second-order logic

In monadic second-order logic in addition to the quantification over individual variables, we can also quantifier over sets of variables, i.e. we can quantifier over properties.

Notation

We use uppercase variables for denoting sets of vertices, and lowercase variables for denoting individual vertices.

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Monadic second-order logic

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Notation

We use uppercase variables for denoting sets of vertices, and lowercase variables for denoting individual vertices.

Example

A graph satisfies the following sentence if only if it is disconnected (Courcelle and Engelfriet 2012):

$$\exists X[\exists x.x \in X \wedge \exists y.y \notin X \wedge \forall x \forall y(\mathrm{edg}(x,y) \Rightarrow (x \in X \Leftrightarrow y \in X)]$$

Undecidable Problems

The r-Neighbourhood Problem

Definition

A **rooted graph** is a graph with a vertex considered as special (see, e.g. Gross, Yellen and P. Zhang (2013)).

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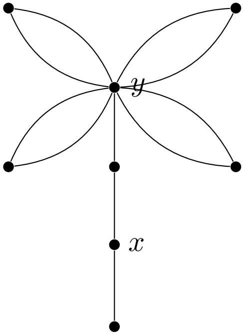
Definition

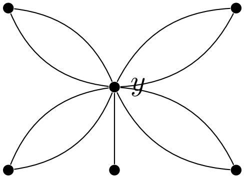
Let G be a graph, v a vertex and k a positive integer. The $\boldsymbol{r}$**-neighbourhood** of v, denoted $N_r(v)$, is the subgraph induced by the set of vertices of distance at most r from v. The graph $N_r(v)$ is a rooted graph with root v.

Example

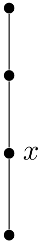
A graph G and two r-neighbourhoods.



G



$N_1(y)$



$N_2(x)$

The r-Neighbourhood Problem

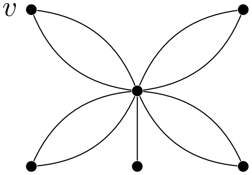
Definition

Let G be a graph. The $\boldsymbol{r}$**-neighbourhood** set of G, denoted $\mathcal{N}_r(G)$, is the set of isomorphism classes of r-neighbourhoods of vertices in G.

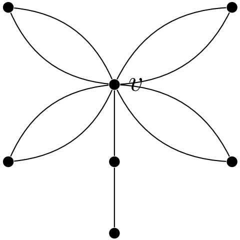
The r-Neighbourhood Problem

Example

For the graph G in this example, the set $\mathcal{N}_2(G)$ consists of the rooted graphs G_1, G_2, G_3, G_4 and G_5.



G_1

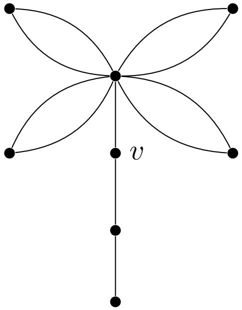


G_2

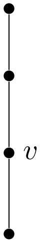
Continued on next slide

The r-Neighbourhood Problem

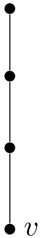
Example (continuation)



G_3



G_4



G_5

The r-Neighbourhood Problem

The r-neighbourhood problem

Given a positive integer k and a finite set Φ of rooted graphs, is there a (connected) graph G whose r-neighbourhood set is Φ?

*Also in a Russian publication translated as V. K. Bulitko (1973), Graphs with Prescribed Environments of the Vertices.

The r-Neighbourhood Problem

The r-neighbourhood problem

Given a positive integer k and a finite set Φ of rooted graphs, is there a (connected) graph G whose r-neighbourhood set is Φ?

Theorem

The r-neighbourhood problem is undecidable (Winkler 1983, Theorem 5).*

*Also in a Russian publication translated as V. K. Bulitko (1973), Graphs with Prescribed Environments of the Vertices.

Undecidable Problems

Some references to other undecidable problems

- Csóka (2012). 'An Undecidability Result on Limits of Sparse Graphs'.
- Jacobs (1994). 'Undecidability of Winkler's r-Neighborhood Problem for Covering Digraphs'.
- Burr (1984). 'Some Undecidable Problems Involving the Edge-Coloring and Vertex-Coloring of Graphs'.
- Foldes and Steinberg (1980). 'A Topological Space for which Graph Embeddability is Undecidable'.

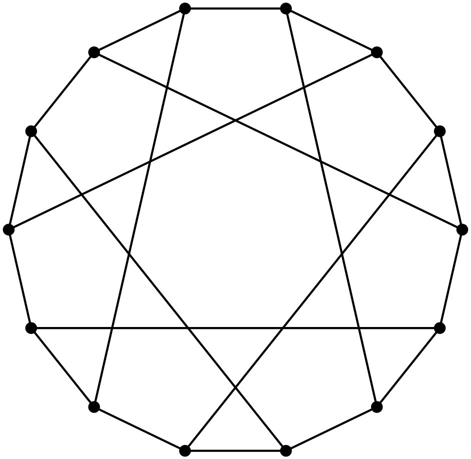
Some Named Graphs

Some Named Graphs

Remark

The graphs on this section were drawn using the `tkz-berge.sty` package by Matthes (2011).

Heawood Graph



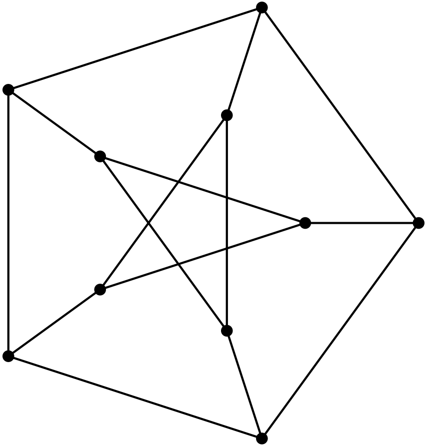
Vertices 14
Edges 21

3-regular

Girth 6

Connected

Petersen Graph



Vertices 10
Edges 15

3-regular

Girth 5

Connected

Appendix: Order Theory

Partially Ordered Sets

Definition

A binary relation $\preceq$ on a set A is a **partial ordering** iff it satisfies the following properties:

$$\forall x(x \preceq x) \qquad \text{(reflexivity)}$$

$$\forall x \forall y(x \preceq y \preceq x \Rightarrow x = y) \qquad \text{(anti-symmetry)}$$

$$\forall x \forall y \forall z(x \preceq y \preceq z \Rightarrow x \preceq z) \qquad \text{(transitivity)}$$

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$$\forall x \forall y \forall z(x \preceq y \preceq z \Rightarrow x \preceq z) \qquad \text{(transitivity)}$$

Definition

Let $\preceq$ be a partial ordering on a set A. The relational structure $(A, \preceq)$ is a **partially ordered set** (or **poset**).

Notable Elements

Let $(A, \preceq)$ be a poset.

Definition

An element $a \in A$ is the **greatest element** (*máximo*) of $(A, \preceq)$ iff $b \preceq a$ for all $b \in A$.

Notable Elements

Let $(A, \preceq)$ be a poset.

Definition

An element $a \in A$ is the **greatest element** (*máximo*) of $(A, \preceq)$ iff $b \preceq a$ for all $b \in A$.

Definition

An element $a \in A$ is the **least element** (*mínimo*) iff $a \preceq b$ for all $b \in A$.

Notable Elements

Let $(A, \preceq)$ be a poset.

Definition

An element $a \in A$ is the **greatest element** (*máximo*) of $(A, \preceq)$ iff $b \preceq a$ for all $b \in A$.

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An element $a \in A$ is the **least element** (*mínimo*) iff $a \preceq b$ for all $b \in A$.

Definition

An element $a \in A$ is a **maximal** of $(A, \preceq)$ iff there is no $b \in A$ such that $a \prec b$.

Notable Elements

Let $(A, \preceq)$ be a poset.

Definition

An element $a \in A$ is the **greatest element** (*máximo*) of $(A, \preceq)$ iff $b \preceq a$ for all $b \in A$.

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An element $a \in A$ is the **least element** (*mínimo*) iff $a \preceq b$ for all $b \in A$.

Definition

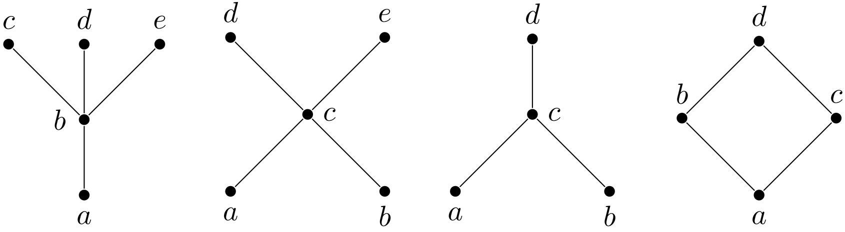
An element $a \in A$ is a **maximal** of $(A, \preceq)$ iff there is no $b \in A$ such that $a \prec b$.

Definition

An element $a \in A$ is a **minimal** $(A, \preceq)$ iff there is no $b \in A$ such that $b \prec a$.

Notable Elements

Example



(a) (b) (c) (d)

Fig.	Least element	Greatest element	Maximals	Minimals
(a)	a		c, d, e	a
(b)			d, e	a, b
(c)		d	d	a, b
(d)	a	d	d	a

Totally Ordered Sets

Definition

A binary relation $\preceq$ on a set A is a **total ordering** iff it satisfies the following properties:

$$\forall x \forall y (x \preceq y \preceq x \Rightarrow x = y) \qquad \text{(anti-symmetry)}$$

$$\forall x \forall y \forall z (x \preceq y \preceq z \Rightarrow x \preceq z) \qquad \text{(transitivity)}$$

$$\forall x \forall y (x \preceq y \vee y \preceq x) \qquad \text{(totality)}$$

Remark

Note that totality implies reflexivity.

Totally Ordered Sets

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$$\forall x \forall y (x \preceq y \vee y \preceq x) \qquad \text{(totality)}$$

Remark

Note that totality implies reflexivity.

Definition

Let $\preceq$ be a total ordering on a set A. The relational structure $(A, \preceq)$ is a **totally ordered set** (also called **linearly ordered set** or **chain**).

Totally Order Sets

Remark

The term **chain** also can refer to a totally ordered subset of some partially ordered set (Vialar 2016).

Appendix: Topology

Topology

Definition

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. A function $f : X \to Y$ is a **homeomorphism** iff:

- the function is a bijection and
- both the function and the inverse function are continues.

That is, $f(U)$ is open if and only if U is open.*

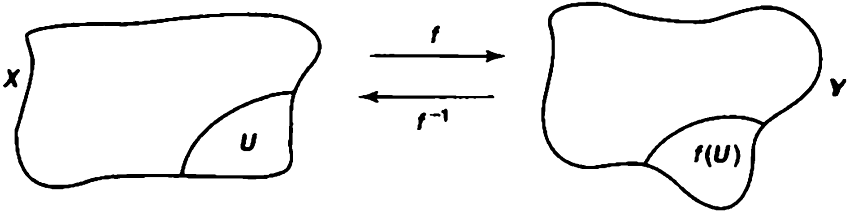

*Figure source: Munkres (2000, Fig. 18.1).

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Invariants Index*

*TODO: The links to page numbers are not working. Tested with TeX Live 2018, `pdfTeX` 3.14159265-2.6-1.40.19, `beamer.cls` v3.50 and `makeindex` v2.15.