CM0845 Logic
Propositional Logic: Natural Deduction

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Preliminaries

Convention

The references for this section are van Dalen [2013, § 2.4 and § 2.6]

Derivation Rules for $\{\wedge, \rightarrow, \perp\}$

Conjunction

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\,\wedge\mathrm{I} \qquad\qquad \frac{\varphi \wedge \psi}{\varphi}\,\wedge\mathrm{E} \qquad\qquad \frac{\varphi \wedge \psi}{\psi}\,\wedge\mathrm{E}$$

Derivation Rules for $\{\wedge, \rightarrow, \perp\}$

Implication

$$\dfrac{\begin{array}{c}[\varphi]^x \\ \vdots \\ \psi\end{array}}{\varphi \rightarrow \psi} \rightarrow\mathrm{I}^x \qquad\qquad \dfrac{\varphi \qquad \varphi \rightarrow \psi}{\psi} \rightarrow\mathrm{E}$$

Remark: In the application of the $\rightarrow\mathrm{I}$ rule, we may discharge zero, one, or more occurrences of the assumption.

Derivation Rules for $\{\wedge, \rightarrow, \perp\}$

Bottom elimination

$$\frac{\perp}{\varphi}\ \perp\mathrm{E}$$

Proof by contradiction (*reductio ad absurdum*)

$$\begin{array}{c}[\neg\varphi]^x \\ \vdots \\ \dfrac{\perp}{\varphi}\ \mathrm{RAA}^x\end{array}$$

where $\neg\varphi := \varphi \rightarrow \perp$.

Derivation Rules for $\{\wedge, \to, \bot\}$

Definition

Let Γ be a set of formulae and let φ be a formula. The relation $\Gamma \vdash \varphi$ means that there is a derivation with conclusion φ from the set of hypotheses Γ.

Example

A derivation where every assumption is discharged once. A proof of Pierce's law $\vdash ((\varphi \to \psi) \to \varphi) \to \varphi$.*

Proof

$$
\begin{prooftree}
\AxiomC{$[\varphi]^x$}
\AxiomC{$[\neg\varphi]^y$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\bot$E}
\UnaryInfC{ψ}
\RightLabel{$\to$I^x}
\UnaryInfC{$\varphi \to \psi$}
\AxiomC{$[(\varphi \to \psi) \to \varphi]^z$}
\RightLabel{$\to$E}
\BinaryInfC{φ}
\AxiomC{$[\neg\varphi]^y$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{RAA^y}
\UnaryInfC{φ}
\RightLabel{$\to$I^z}
\UnaryInfC{$(\varphi \to \psi) \to \varphi) \to \varphi$}
\end{prooftree}
$$

*Adapted from [Alastair 2017].

Example

A derivation using the same assumption twice. A proof that $\vdash (\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$.

Proof

$$\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]^x}{\psi}\,\wedge\mathrm{E} \qquad \dfrac{[\varphi \wedge \psi]^x}{\varphi}\,\wedge\mathrm{E}}{\psi \wedge \varphi}\,\wedge\mathrm{I}}{(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)}\,\rightarrow\mathrm{I}^x$$

Derivation Rules for $\{\wedge, \rightarrow, \perp\}$

Example

A derivation where the assumption and the conclusion are the same. A proof that $\vdash \varphi \rightarrow \varphi$.

Proof

$$\frac{[\varphi]^x}{\varphi \rightarrow \varphi} \rightarrow\mathrm{I}^x$$

Derivation Rules for $\{\wedge, \rightarrow, \bot\}$

Remark

'The rule schemes of natural deduction display only the open assumptions that are **active** in the rule, but there may be any number of other assumptions.' [Negri and von Plato 2008, p. 10]

Derivation Rules for $\{\wedge, \rightarrow, \perp\}$

Remark

'The rule schemes of natural deduction display only the open assumptions that are **active** in the rule, but there may be any number of other assumptions.' [Negri and von Plato 2008, p. 10]

Example

A derivation where there is a vacuous discharge when using the inference rule $\rightarrow$I. A proof that $\vdash \varphi \rightarrow (\psi \rightarrow \varphi)$.

Proof

$$\dfrac{\dfrac{[\varphi]^x}{\psi \rightarrow \varphi}\ {\rightarrow}\mathrm{I}\ \text{(vacuous discharge of } \psi\text{)}}{\varphi \rightarrow (\psi \rightarrow \varphi)}\ {\rightarrow}\mathrm{I}^x$$

Derivation Rules for $\{\wedge, \rightarrow, \bot\}$

Example

A derivation using one hypothesis. A proof that $\varphi \vdash \neg(\neg\varphi \wedge \psi)$ [van Dalen 2013, Exercise 3.(a), p. 37].

Proof

$$\dfrac{\dfrac{\varphi \qquad \dfrac{[\neg\varphi \wedge \psi]^x}{\neg\varphi}\,\wedge\mathrm{E}}{\bot}\,\rightarrow\mathrm{E}}{\neg(\neg\varphi \wedge \psi)}\,\rightarrow\mathrm{I}^x$$

Derivation Rules for $\{\wedge, \to, \bot\}$

Example

A derivation using the same hypothesis twice. A proof that $\varphi \wedge \psi \vdash \psi \wedge \varphi$.

Proof

$$\dfrac{\dfrac{\varphi \wedge \psi}{\psi}\,\wedge\mathrm{E} \qquad \dfrac{\varphi \wedge \psi}{\varphi}\,\wedge\mathrm{E}}{\psi \wedge \varphi}\,\wedge\mathrm{I}$$

Set of Derivations

Notation

(Whiteboard)

Definition (van Dalen [2013], Definition 2.4.1)

The **set of derivations**, denoted $\mathcal{D}$, is the smallest set X with the properties:

(see next slide)

Set of Derivations

(1) The one-element tree φ belongs to X for all $\varphi \in PROP$.

(2$\wedge$) If $\begin{array}{c}\mathcal{D}\\ \varphi\end{array}, \begin{array}{c}\mathcal{D}'\\ \varphi'\end{array} \in X$, then $\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}'\\ \varphi & \varphi'\end{array}}{\varphi\wedge\varphi'} \in X$.

If $\begin{array}{c}\mathcal{D}\\ \varphi\wedge\psi\end{array} \in X$, then $\dfrac{\begin{array}{c}\mathcal{D}\\ \varphi\wedge\psi\end{array}}{\varphi}, \dfrac{\begin{array}{c}\mathcal{D}\\ \varphi\wedge\psi\end{array}}{\psi} \in X$.

(2$\rightarrow$) If $\begin{array}{c}\varphi\\ \mathcal{D}\\ \psi\end{array} \in X$, then $\dfrac{\begin{array}{c}[\varphi]\\ \mathcal{D}\\ \psi\end{array}}{\varphi\rightarrow\psi} \in X$.

If $\begin{array}{c}\mathcal{D}\\ \varphi\end{array}, \begin{array}{c}\mathcal{D}'\\ \varphi\rightarrow\psi\end{array} \in X$, then $\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}'\\ \varphi & \varphi\rightarrow\psi\end{array}}{\psi} \in X$.

(2$\bot$) If $\begin{array}{c}\mathcal{D}\\ \bot\end{array} \in X$, then $\dfrac{\begin{array}{c}\mathcal{D}\\ \bot\end{array}}{\varphi} \in X$.

If $\begin{array}{c}\neg\varphi\\ \mathcal{D}\\ \bot\end{array} \in X$, then $\dfrac{\begin{array}{c}[\neg\varphi]\\ \mathcal{D}\\ \bot\end{array}}{\varphi} \in X$.

Disjunction

$$\frac{\varphi}{\varphi \vee \psi}\,\vee\mathrm{I} \qquad \frac{\psi}{\varphi \vee \psi}\,\vee\mathrm{I} \qquad \frac{\varphi \vee \psi \quad \begin{array}{c}[\varphi]^x \\ \vdots \\ \sigma\end{array} \quad \begin{array}{c}[\psi]^y \\ \vdots \\ \sigma\end{array}}{\sigma}\,\vee\mathrm{E}^{x,y}$$

Derivation Rules for the Missing Connectives $\{\vee, \neg, \leftrightarrow\}$

Negation

$$\cfrac{\begin{array}{c}[\varphi]^x \\ \vdots \\ \bot\end{array}}{\neg\varphi}\ \neg\mathrm{I}^x \qquad\qquad \cfrac{\varphi \qquad \neg\varphi}{\bot}\ \neg\mathrm{E}$$

Derivation Rules for the Missing Connectives $\{\vee, \neg, \leftrightarrow\}$

Equivalence

$$\dfrac{\begin{matrix}[\varphi]^x \\ \vdots \\ \psi\end{matrix} \qquad \begin{matrix}[\psi]^y \\ \vdots \\ \varphi\end{matrix}}{\varphi \leftrightarrow \psi} \leftrightarrow\mathrm{I}^{\mathrm{x,y}} \qquad\qquad \dfrac{\varphi \qquad \varphi \leftrightarrow \psi}{\psi} \leftrightarrow\mathrm{E} \qquad\qquad \dfrac{\psi \qquad \varphi \leftrightarrow \psi}{\varphi} \leftrightarrow\mathrm{E}$$

Derivation Rules for $\{\wedge, \vee, \rightarrow, \bot\}$

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\ \wedge\mathrm{I} \qquad\qquad \frac{\varphi \wedge \psi}{\varphi}\ \wedge\mathrm{E} \qquad\qquad \frac{\varphi \wedge \psi}{\psi}\ \wedge\mathrm{E}$$

$$\frac{\varphi}{\varphi \vee \psi}\ \vee\mathrm{I} \qquad\qquad \frac{\psi}{\varphi \vee \psi}\ \vee\mathrm{I} \qquad\qquad \begin{array}{c} \begin{array}{ccc} & [\varphi]^x & [\psi]^y \\ & \vdots & \vdots \\ \varphi \vee \psi & \sigma & \sigma \end{array} \\ \hline \sigma \end{array}\ \vee\mathrm{E}^{x,y}$$

$$\begin{array}{c} [\varphi]^x \\ \vdots \\ \psi \\ \hline \varphi \rightarrow \psi \end{array}\ \rightarrow\mathrm{I}^x \qquad\qquad \frac{\varphi \qquad \varphi \rightarrow \psi}{\psi}\ \rightarrow\mathrm{E}$$

$$\frac{\bot}{\varphi}\ \bot\mathrm{E} \qquad\qquad \begin{array}{c} [\neg\varphi]^x \\ \vdots \\ \bot \\ \hline \varphi \end{array}\ \mathrm{RAA}^x$$

Example

Prove that $\vdash \varphi \vee \neg\varphi$ [van Dalen 2013, example p. 49].

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^x}{\varphi \vee \neg\varphi}\,\vee\mathrm{I} \qquad [\neg(\varphi \vee \neg\varphi)]^y}{\bot}\,\to\mathrm{E}}{\neg\varphi}\,\to\mathrm{I}^x}{\varphi \vee \neg\varphi}\,\vee\mathrm{I} \qquad [\neg(\varphi \vee \neg\varphi)]^y}{\bot}\,\to\mathrm{E}}{\varphi \vee \neg\varphi}\,\mathrm{RAA}^y$$

Natural Deduction in Sequent Calculus Style

$$\frac{}{\Gamma, \varphi \vdash \varphi}\ \mathrm{Ax}$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}\ \wedge\mathrm{I} \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}\ \wedge\mathrm{E} \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}\ \wedge\mathrm{E}$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ \vee\mathrm{I} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ \vee\mathrm{I} \qquad \frac{\Gamma \vdash \varphi \vee \psi \qquad \Gamma, \varphi \vdash \sigma \qquad \Gamma, \psi \vdash \sigma}{\Gamma \vdash \sigma}\ \vee\mathrm{E}$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}\ {\to}\mathrm{I} \qquad \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi \to \psi}{\Gamma \vdash \psi}\ {\to}\mathrm{E}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\ \bot\mathrm{E} \qquad \frac{\Gamma, \neg\varphi \vdash \bot}{\Gamma \vdash \varphi}\ \mathrm{RAA}$$

Example

We prove that $\vdash \varphi \lor \neg\varphi$.

(continued on next slide)

Proof

Let $\Gamma = \{\varphi, \neg(\varphi \vee \neg\varphi)\}$ and $\Delta = \Gamma - \{\varphi\}$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \varphi}\,\text{Ax}}{\Gamma \vdash \varphi \vee \neg\varphi}\,\vee\text{I} \qquad \dfrac{}{\Gamma \vdash \neg(\varphi \vee \neg\varphi)}\,\text{Ax}}{\Gamma \vdash \bot}\,\to\text{E}}{\Delta \vdash \neg\varphi}\,\to\text{I}}{\Delta \vdash \varphi \vee \neg\varphi}\,\vee\text{I} \qquad \dfrac{}{\Delta \vdash \neg(\varphi \vee \neg\varphi)}\,\text{Ax}}{\dfrac{\Delta \vdash \bot}{\vdash \varphi \vee \neg\varphi}\,\text{RAA}}\,\to\text{E}
$$

Example

A derivation where there is a vacuous discharge when using the inference rule $\rightarrow$I. A proof that $\vdash \varphi \rightarrow (\psi \rightarrow \varphi)$.

Proof

$$\dfrac{\dfrac{\dfrac{}{\varphi \vdash \varphi}\,\mathrm{Ax}}{\varphi \vdash \psi \rightarrow \varphi}\,{\rightarrow}\mathrm{I}\ \text{(vacuous discharge of } \psi\text{)}}{\vdash \varphi \rightarrow (\psi \rightarrow \varphi)}\,{\rightarrow}\mathrm{I}$$

References

- Alastair, Carr (2017). Natural Deduction Pack. URL: https://github.com/Alastair-Carr/Natural-Deduction-Pack (visited on 22/07/2017) (cit. on p. 7).
- Negri, Sara and von Plato, Jan [2001] (2008). Structural Proof Theory. Digitally printed version. Cambridge University Press (cit. on pp. 10, 11).
- van Dalen, Dirk [1980] (2013). Logic and Structure. 5th ed. Springer (cit. on pp. 2, 12, 14, 20).