

CM0845 Logic

First-Order Logic: Natural Deduction

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Remark

The references for this section are van Dalen [2013, § 3.8 and § 3.9].

Derivation Rules for the Quantifiers

Universal quantifier

$$\frac{\varphi(x)}{\forall x\varphi(x)} \forall\text{I}$$

$$\frac{\forall x\varphi(x)}{\varphi[t/x]} \forall\text{E}$$

Side condition: In $\forall\text{I}$, the variable x may not occur free in any hypothesis on which $\varphi(x)$ depends.

Derivation Rules for the Quantifiers

Existential quantifier

$$\frac{\varphi[t/x]}{\exists x\varphi(x)} \exists\text{I}$$

$$\frac{\begin{array}{c} [\varphi(x)]^x \\ \vdots \\ \exists x\varphi(x) \end{array} \quad \psi}{\psi} \exists\text{E}^x$$

Side condition: In $\exists\text{E}$, the variable x is not free in ψ , or in a hypothesis of the sub-derivation of ψ , other than $\varphi(x)$.

Derivation Rules for the Quantifiers

Example

The derivation rules for the quantifiers are consistent with the convention that the universe of discourse is not empty, so we can prove that $\vdash \forall x\varphi(x) \rightarrow \exists x\varphi(x)$.

Proof

$$\frac{\frac{\frac{[\forall x\varphi(x)]^x}{\varphi(x)} \forall E}{\exists x\varphi(x)} \exists I}{\forall x\varphi(x) \rightarrow \exists x\varphi(x)} \rightarrow I^x$$

Derivation Rules for the Quantifiers

Example

Prove that $\vdash \exists x(\varphi(x) \vee \psi(x)) \rightarrow \exists x\varphi(x) \vee \exists x\psi(x)$ [van Dalen 2013, p. 92].

Proof

$$\frac{\frac{\frac{[\varphi(x)]^x}{\exists x\varphi(x)} \exists\text{I} \quad \frac{[\psi(x)]^y}{\exists x\psi(x)} \exists\text{I}}{\frac{[\varphi(x) \vee \psi(x)]^z \exists x\varphi(x) \vee \exists x\psi(x)}{\exists x\varphi(x) \vee \exists x\psi(x)} \vee\text{I}} \vee\text{I} \quad \frac{[\varphi(x) \vee \psi(x)]^z \exists x\varphi(x) \vee \exists x\psi(x)}{\exists x\varphi(x) \vee \exists x\psi(x)} \vee\text{E}^{x,y}}{\frac{[\exists x(\varphi(x) \vee \psi(x))]^w \quad \exists x\varphi(x) \vee \exists x\psi(x)}{\exists x\varphi(x) \vee \exists x\psi(x)} \exists\text{E}^z} \rightarrow\text{I}^w$$

Derivation Rules for the Identity

Identity

$$\frac{}{x = x} \text{RI}_1$$

$$\frac{x = y}{y = x} \text{RI}_2$$

$$\frac{x = y \quad y = z}{x = z} \text{RI}_3$$

$$\frac{x_1 = y_1, \dots, x_n = y_n}{t[x_1, \dots, x_n/z_1, \dots, z_n] = t[y_1, \dots, y_n/z_1, \dots, z_n]} \text{RI}_4$$

$$\frac{x_1 = y_1, \dots, x_n = y_n \quad \varphi[x_1, \dots, x_n/z_1, \dots, z_n]}{\varphi[y_1, \dots, y_n/z_1, \dots, z_n]} \text{RI}_4$$

Derivation Rules for the Identity

Example

The derivation rules for the identity are consistent with the convention that the universe of discourse is not empty, so we can prove that $\vdash \exists x(x = x)$.

Proof

$$\frac{\frac{}{x = x} \text{RI}_1}{\exists x(x = x)} \exists\text{I}$$

References



van Dalen, Dirk [1980] (2013). Logic and Structure. 5th ed. Springer (cit. on pp. 2, 6).