CM0832 Elements of Set Theory
3. Relations and Functions

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Enderton 1977].

Ordered Pairs

Remark

Let a and b be sets. An ordered pair $\langle a, b\rangle$ should be a set such that

$$\langle a, b\rangle = \langle c, d\rangle \quad \text{iff} \quad a = c \wedge b = d.$$

Definition

We define an **ordered pair** using Kuratowski's definition, that is,

$$\langle a, b\rangle := \{\{a\}, \{a, b\}\}.$$

Ordered Pairs

Example

We show that $\langle \emptyset, \{\emptyset\} \rangle \neq \langle \{\emptyset\}, \emptyset \rangle$.

$$\begin{aligned}
\langle \emptyset, \{\emptyset\} \rangle &= \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\
&= \{\{\emptyset\}, \{\{\emptyset\}, \emptyset\}\} \\
&\neq \{\{\{\emptyset\}\}, \{\{\emptyset\}, \emptyset\}\} \\
&= \langle \{\emptyset\}, \emptyset \rangle.
\end{aligned}$$

Ordered Pairs

Example

Let a be a set. Then

$$\begin{aligned}\langle a, a\rangle &= \{\{a\}, \{a, a\}\}\\ &= \{\{a\}, \{a\}\}\\ &= \{\{a\}\}.\end{aligned}$$

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Exercise

To give a different definition of ordered pair.

Cartesian Product

Definition

Let A and B be sets. The **Cartesian product** of A and B is defined by

$$A \times B := \{\, \langle x, y \rangle \mid x \in A \wedge y \in B \,\}.$$

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Let A and B be sets. Note that $A \times B$ is a set because we can define it via the subset axiom scheme.

$$A \times B := \{\, \langle x, y \rangle \in \mathcal{P}\mathcal{P}(A \cup B) \mid x \in A \wedge y \in B \,\}.$$

Relations

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Let R the relation defined by $R = \{\langle a, b \rangle, \langle b, b \rangle, \langle c, b \rangle\}$. Diagram: whiteboard.

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Let R the relation defined by $R = \{\langle a, b\rangle, \langle b, b\rangle, \langle c, b\rangle\}$. Diagram: whiteboard.

Example

Let $\omega = \{0, 1, 2, \ldots\}$. The identity relation on ω is defined by

$$\begin{aligned} \mathrm{I}_\omega &:= \{\, \langle n, n\rangle \mid n \in \omega \,\} \\ &= \{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle, \ldots\}. \end{aligned}$$

Relations

Definition

Let R be a relation. We define the **domain**, the **range** and the **field** of R by

$$\begin{aligned} \operatorname{dom} R &:= \{\, x \mid \exists y(\langle x, y\rangle \in R) \,\}, \\ \operatorname{ran} R &:= \{\, y \mid \exists x(\langle x, y\rangle \in R) \,\}, \\ \operatorname{fld} R &:= \operatorname{dom} R \cup \operatorname{ran} R. \end{aligned}$$

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Remark

Let R be a relation. Note that $\operatorname{dom} R$ and $\operatorname{ran} R$ are sets because we can define them via the subset axiom scheme.

$$\begin{aligned} \operatorname{dom} R &:= \left\{\, x \in \bigcup\bigcup R \;\middle|\; \exists y(\langle x, y\rangle \in R) \,\right\}, \\ \operatorname{ran} R &:= \left\{\, y \in \bigcup\bigcup R \;\middle|\; \exists x(\langle x, y\rangle \in R) \,\right\}. \end{aligned}$$

n-Ary Relations

Definition

We define an **ordered n-tuple**, for $n \geq 3$, by

$$\langle x_1, x_2, \ldots, x_n \rangle := \langle \langle x_1, x_2, \ldots, x_{n-1} \rangle, x_n \rangle$$

n-Ary Relations

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Example

Ordered triple (3-tuple) and ordered quadruple (4-tuple).

$$\langle x_1, x_2, x_3 \rangle := \langle \langle x_1, x_2 \rangle, x_3 \rangle,$$
$$\langle x_1, x_2, x_3, x_4 \rangle := \langle \langle x_1, x_2, x_3 \rangle, x_4 \rangle.$$

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$$\langle x_1, x_2, x_3, x_4 \rangle := \langle \langle x_1, x_2, x_3 \rangle, x_4 \rangle.$$

Definition

We define an 1-**tuple** by

$$\langle x \rangle := x.$$

n-Ary Relations

Definition

Let A be a set. We define an **n-ary relation on A** to be a set of ordered n-tuples with all components in A.

n-Ary Relations

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Example

Whiteboard.

Remark

Let A be a set. Note that an 1-ary relation on A is just a subset of A but it is not a relation.

Functions

Definition

A **function** (**mapping** or **correspondence**) is a relation F such that for each x in $\operatorname{dom} F$ there is only one y such that xFy.

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Notation

We write $F : A \to B$ iff F is a function, $\operatorname{dom} F = A$ and $\operatorname{ran} F \subseteq B$.

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Notation

We write $F : A \to B$ iff F is a function, $\operatorname{dom} F = A$ and $\operatorname{ran} F \subseteq B$.

Definition

Let F be a function and A and B sets.

(i) F is a function **on** (**from**) A iff $\operatorname{dom} F = A$.

(ii) F is a function **into** (**to**) B iff $\operatorname{ran} F \subseteq B$.

(iii) F is a function **onto** B iff $\operatorname{ran} F = B$.

Functions

Exercise 3.11

Prove the following version (for functions) of the extensionality principle: Assume that F and G are functions, $\operatorname{dom} F = \operatorname{dom} G$, and $F(x) = G(x)$ for all x in the common domain. Then $F = G$.

Functions

Definition

A function F is **one-to-one** (or **injective**) iff for each $y \in \operatorname{ran} F$ there is only one x such that xFy. In other words, if $x_1, x_2 \in \operatorname{dom} F$ and $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

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Functions

Definition

Let A, F and G be sets. We define, the **inverse** of F, the **composition** of F and G, the **restriction** of F to A and the **image** of A under F by

$$F^{-1} := \{ \langle y, x \rangle \mid xFy \} \qquad \text{(inverse of } F\text{)}$$

$$F \circ G := \{ \langle x, y \rangle \mid \exists t\, (xGt \wedge tFy) \} \qquad \text{(composition of } F \text{ and } G\text{)}$$

$$F \restriction A := \{ \langle x, y \rangle \mid x \in A \wedge xFy \} \qquad \text{(restriction of } F \text{ to } A\text{)}$$

$$\begin{aligned} F[\![A]\!] &:= \operatorname{ran}(F \restriction A) \qquad \text{(image of } A \text{ under } F\text{)} \\ &= \{ y \mid \exists x\, (x \in A \wedge xFy) \} \end{aligned}$$

Functions

Example

Let

$$F = \{\langle \emptyset, a\rangle, \langle \{\emptyset\}, b\rangle\}.$$

Then

$$\begin{aligned} \operatorname{dom} F &= \{\emptyset, \{\emptyset\}\} \\ \operatorname{ran} F &= \{a, b\}, \\ F^{-1} &= \{\langle a, \emptyset\rangle, \langle b, \{\emptyset\}\rangle\}, \\ F \restriction \emptyset &= \emptyset, \\ F \restriction \{\emptyset\} &= \{\langle \emptyset, a\rangle\}, \\ F[\![\{\emptyset\}]\!] &= \{a\}, \\ F(\{\emptyset\}) &= b. \end{aligned}$$

F is a function,

F^{-1} is function iff $a \neq b$,

Functions

Exercise 3.18

Let R be the set

$$\{\langle 0,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}.$$

To find $R \circ R$, $R \restriction \{1\}$, $R^{-1} \restriction \{1\}$, $R[\![\{1\}]\!]$ and $R^{-1}[\![\{1\}]\!]$.

Functions

Exercise 3.18

Let R be the set

$$\{\langle 0,1\rangle,\langle 0,2\rangle,\langle 0,3\rangle,\langle 1,2\rangle,\langle 1,3\rangle,\langle 2,3\rangle\}.$$

To find $R\circ R$, $R\restriction\{1\}$, $R^{-1}\restriction\{1\}$, $R[\![\{1\}]\!]$ and $R^{-1}[\![\{1\}]\!]$.

Exercise (p. 44)

Let A, F and G be sets. Show that F^{-1}, $F\circ G$, $F\restriction A$ and $F[\![A]\!]$ are sets.

Functions

Theorem 3E

Let F be a set. Then

$$\operatorname{dom} F^{-1} = \operatorname{ran} F \quad \text{and} \quad \operatorname{ran} F^{-1} = \operatorname{dom} F.$$

If additionally F is a relation, then

$$(F^{-1})^{-1} = F.$$

Functions

Theorem 3G

Let F be an one-to-one function.

- If $x \in \operatorname{dom} F$, then
$$F^{-1}(F(x)) = x.$$
- If $y \in \operatorname{ran} F$, then
$$F(F^{-1}(y)) = y.$$

Functions

Theorem 3H

Let F and G be functions. Then

- $F \circ G$ is a function,
- $\operatorname{dom}(F \circ G) = \{\, x \in \operatorname{dom} G \mid G(x) \in \operatorname{dom} F \,\}$ and
- if $x \in \operatorname{dom}(F \circ G)$, then $(F \circ G)(x) = F(G(x))$.

Functions

Theorem 3I

Let F and G be sets. Then

$$(F \circ G)^{-1} = G^{-1} \circ F^{-1}.$$

Functions

Theorem 3J

Let F be a function $F : A \to B$ and $A \neq \emptyset$.

(i) There exists a function $G : B \to A$ (a "left inverse") such that $G \circ F$ is the identity function I_A on A iff the function F is one-to-one.

(ii) There exists a function $H : B \to A$ (a "right inverse") such that $F \circ H$ is the identity function I_B on B iff the function F maps A onto B.

Functions

Axiom of choice (first form)

For any relation R there is a function $H \subseteq R$ with $\operatorname{dom} H = \operatorname{dom} R$.

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Example

Whiteboard.

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Example

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Remark

Is the axiom of choice accepted in constructive mathematics? (See, e.g. Martin-Löf [2006]).

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Definition

Let A and B be sets. We define the **set of functions** from A into B by

$$B^A := \{ F \mid F : A \to B \} =: {}^A B.$$

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Example

- $\{0,1\}^\omega$: The set of infinity binary sequences.

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- $\emptyset^A = \emptyset$ for $A \neq \emptyset$ (no function can have a non-empty domain and an empty range).

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Example

- $\{0, 1\}^\omega$: The set of infinity binary sequences.
- $\emptyset^A = \emptyset$ for $A \neq \emptyset$ (no function can have a non-empty domain and an empty range).
- $A^\emptyset = \{\emptyset\}$ for any set A ($\emptyset$ is the only function with an empty domain).

Functions

Remark

Let A and B be sets. Note that B^A is a set because we can define it via the subset axiom scheme.

$$B^A := \{ F \in \mathcal{P}(A \times B) \mid F : A \to B \}.$$

Families

Remark

Families is another way to express functions when the range of a function is more important than the function itself. We write functions as families when we want to put the emphasis on the values of the function rather in the function.*

*Enderton [1977] do not use families, but 'only' functions.

Families

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Families is another way to express functions when the range of a function is more important than the function itself. We write functions as families when we want to put the emphasis on the values of the function rather in the function.*

Remark

The terminology and notation on families is not established.

*Enderton [1977] do not use families, but 'only' functions.

Families

Definition

Let I and X be sets. A **family in X indexed by I** is a function

$$A : I \to X$$
$$A = \{ \langle i, A_i \rangle \mid i \in I \text{ and } A_i \in X \},$$

where $A_i := A(i)$, for all $i \in I$.* The set I is the **index set** of the family.

*See, e.g. Halmos [1960], Drake [1974] and Hamilton [(1982) 1992].

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Notation

The above family A is denoted by $\langle A_i \mid i \in I \rangle$ following to [Hrbacek and Jech (1978) 1999].

*See, e.g. Halmos [1960], Drake [1974] and Hamilton [(1982) 1992].

Families

Definition

The **union** of a family $\langle A_i \mid i \in I \rangle$ is defined by

$$\bigcup_{i \in I} A_i := \bigcup \{ A_i \mid i \in I \}$$
$$= \{ x \mid x \in A_i \text{ for some } i \text{ in } I \}.$$

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Example

Whiteboard.

Families

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The **intersection** of a family $\langle A_i \mid i \in I \rangle$ is defined by

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Example

Whiteboard.

Families

Definition

The **Cartesian product** (or **generalised product**) of a family $\langle A_i \mid i \in I \rangle$ is defined by

$$\bigtimes_{i \in I} A_i := \{\, f \mid f : I \to \bigcup_{i \in I} A_i \text{ and } \forall i \, (i \in I \to f(i) \in A_i) \,\} =: \prod_{i \in I} A_i .$$

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Example

Let $\langle A_i \mid i \in I \rangle$ be a family. If $A_i = B$ for all $i \in I$, then

$$\begin{aligned} \bigtimes_{i \in I} A_i &= B^I \\ &= \{\, f \mid f : I \to B \,\}. \end{aligned}$$

Families

Example

The following example illustrates the generalisation of the Cartesian product.

Let X and Y be two sets. Recall that the Cartesian product of X and Y was defined by

$$X \times Y := \{ \langle x, y \rangle \mid x \in X \land y \in Y \}.$$

(continued on next slide)

Families

Example (continuation)

Let $I = \{a, b\}$ be an index set and let $\langle Z_i \mid i \in I \rangle$ be a family where $Z_a = X$ and $Z_b = Y$. Then

$$\bigtimes_{i \in I} Z_i = \{\, f \mid f : I \to X \cup Y, \text{ such that } f(a) \in X \text{ and } f(b) \in Y \,\}.$$

Now, we can define the one-to-one correspondence

$$h : \bigtimes_{i \in I} Z_i \to X \times Y$$
$$h(f) = \langle f(a), f(b) \rangle.$$

Families

Axiom of choice (second form)

Let $\langle H_i \mid i \in I \rangle$ be a family. If $H(i) \neq \emptyset$ for all $i \in I$, then $\bigtimes_{i \in I} H(i) \neq \emptyset$.*

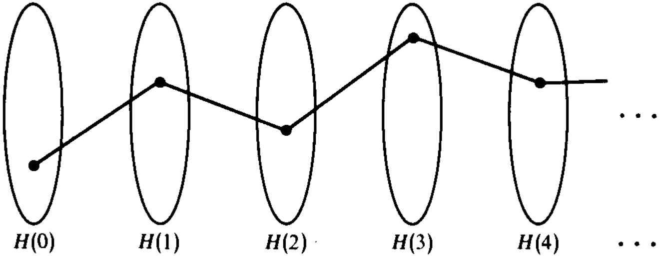

*Figure source: Enderton [1977, Fig. 11].

Equivalence Relations

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- **reflexive** iff xRx for all $x \in A$,
- **symmetric** iff xRy implies yRx for all $x, y \in A$ and
- **transitive** iff xRy and yRz imply xRz for all $x, y, z \in A$.

Example

Whiteboard.

Equivalence Relations

Introduction

Whiteboard.

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Definition

Let R be a binary relation on a set A. The relation R is an **equivalence relation** iff R is reflexive, symmetric and transitive.

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Equivalence Relations

Questions

- Let $A = \{a, e, i, o, u\}$. Is the equality relation on A an equivalence relation?

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Equivalence Relations

Questions

- Let $A = \{a, e, i, o, u\}$. Is the equality relation on A an equivalence relation?
- Let $A \neq \emptyset$ be a set. Is the relation $\emptyset$ on A an equivalence relation?
- Let A be a set. Is the relation $A \times A$ an equivalence relations?
- Let A be a singleton. It is possible to define an equivalence relation on A?

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Definition

Let R be an equivalence relation on a set A and let $x \in \operatorname{fld} R$. The set $[x]_R$ is the **equivalence class of x (modulo R)**.

Notation

We write $[x]$ if the relation R is clear in the context.

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Example

Whiteboard.

Equivalence Relations

Theorem 3N

Let R be an equivalence relation on a set A and let $x, y \in A$. Then

$$[x]_R = [y]_R \quad \text{iff} \quad xRy.$$

Equivalence Relations

Theorem 3P

Let R be an equivalence relation on a set A. Then the set

$$\{ [x]_R \mid x \in A \}$$

of all equivalence classes is a partition of the set A.

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Exercise 3.37

Assume that Π is a partition of a set A. Define the relation R_Π as follows:

$$xR_\Pi y \quad \text{iff} \quad (\exists B \in \Pi)(x \in B \land y \in B).$$

Show that R_Π is an equivalence relation on A.

Equivalence Relations

Definition

Let R be an equivalence relation on a set A. The **quotient set** is defined by

$$A/R := \{ [x]_R \mid x \in A \}.$$

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Remark

Using the λ-notation we could define the natural map by the anonymous function $\lambda x.[x]_R$.

Linear Ordering Relations

Motivation

What means that R is an ordering relation on a set A?

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What means that R is an ordering relation on a set A?

Definition

Let R be a binary relation on a set A. The relation R satisfies **trichotomy** if exactly one of the three alternatives

$$xRy, \quad x = y \quad \text{or} \quad yRx$$

holds for all $x, y \in A$.

Linear Ordering Relations

Definition

Let A be a set. A **linear ordering** (or **total ordering**) on A is a binary relation R on A such that:

(i) R is transitive relation and

(ii) R satisfies trichotomy.

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(i) R is transitive relation and

(ii) R satisfies trichotomy.

Example

References

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