CM0832 Elements of Set Theory
Natural Numbers

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Hrbacek and Jech (1978) 1999].

Defining the Natural Numbers

Approaches for introducing mathematical objects

- Axiomatic
- Definitional

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Question

How to define natural numbers in terms of sets?

Inductive Sets

von Neumann's construction

Informal idea: A natural number is the set of all smaller natural numbers

$$
\begin{aligned}
0 &:= \emptyset, \\
1 &:= \{0\} &&= \{\emptyset\}, \\
2 &:= \{0, 1\} &&= \{\emptyset, \{\emptyset\}\}, \\
3 &:= \{0, 1, 2\} &&= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\vdots
\end{aligned}
$$

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Some 'extra' properties

$$0 \in 1 \in 2 \in 3 \in \cdots \quad \text{and} \quad 0 \subseteq 1 \subseteq 2 \subseteq 3 \subseteq \cdots .$$

Inductive Sets

A wrong impredicative definition

$$n := \{0, 1, \ldots, n-1\}.$$

> *'We cannot just say that a set n is a natural number if its elements are all the smaller natural numbers, because such a "definition" would involve the very concept being defined.' (Hrbacek and Jech [1978] 1999, p. 40)*

Inductive Sets

Definition

Let a be a set. The **successor** of a is

$$\mathrm{S}(a) := a \cup \{a\}.$$

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Example

$$\begin{aligned}
0 &= \emptyset, \\
1 &= S(\emptyset), \\
2 &= S(S(\emptyset)), \\
3 &= S(S(S(\emptyset))), \\
&\vdots
\end{aligned}$$

Inductive Sets

Definition

A set I is **inductive** iff

i) $\emptyset \in I$ and

ii) if $n \in I$ then $\mathrm{S}(n) \in I$.

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Observation

An inductive set will be an infinite set.

The Set of Natural Numbers

Definition

The set of **all natural numbers**, denoted by $\mathbf{N}$, is defined by

$$\mathbf{N} := \{\, x \mid x \in I \text{ for every inductive set } I \,\}.$$

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Let A be an inductive set. For using the Axiom of Comprehension we defined the set of all natural numbers by

$$\mathbf{N} := \{\, x \in A \mid x \in I \text{ for every inductive set } I \,\}.$$

The Set of Natural Numbers

Definition

A **natural number** is a element of $\mathbf{N}$, that is, a set x is a natural number iff x belongs to every inductive set.

The Set of Natural Numbers

Question

Are there inductive sets?

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Observation

So far we only have the set $\emptyset$ and the axioms have the form: For every set X, there exists a set Y such that ....

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Observation

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The Axiom of Infinity

There exists an inductive set.

$$\exists A \, [\, \emptyset \in A \wedge \forall a \, (a \in A \to \mathrm{S}(a) \in A) \,].$$

The Set of Natural Numbers

Theorem (Enderton [1977, Theorem 4B])

The set $\mathbf{N}$ is inductive, and it is a subset of every other inductive set.

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Observation

The set $\mathbf{N}$ is the smallest inductive set

Defining Natural Numbers as Sets

Observation

So far, we defined natural numbers on terms of sets. A different point of view is stated by some authors (see, e.g. Benacerraf [1965]).

Ordering on Natural Numbers

Definition

We define the relation $<$ on $\mathbf{N}$ by

$$m < n \quad \text{iff} \quad m \in n.$$

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Theorem

The pair $(\mathbf{N}, <)$ is a linearly ordered set.

Transitive Sets

Definition

Let A be a set. The set A is a **transitive set** iff every member of a member of A is itself a member of A, that is,

$$x \in a \in A \quad \text{implies} \quad x \in A.$$

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Example

Whiteboard.

Transitive Sets

Theorem

A set A is a transitive set iff $\bigcup A \subseteq A$.

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A set A is a transitive set iff $\bigcup A \subseteq A$.

Proof (only if).

Let A be a transitive set. Then

$$
\begin{aligned}
x \in \bigcup A &\Rightarrow \exists b\,(x \in b \wedge b \in A) && \text{(by definition of } \textstyle\bigcup A\text{)} \\
&\Rightarrow x \in A && \text{(because } A \text{ is transitive)}
\end{aligned}
$$

■

Proof (if).

Let $\bigcup A \subseteq A$. Then

$$
\begin{aligned}
x \in a \wedge a \in A &\Rightarrow x \in \bigcup A && \text{(by definition of } \textstyle\bigcup A\text{)} \\
&\Rightarrow x \in A && \text{(because } \textstyle\bigcup A \subseteq A\text{)}
\end{aligned}
$$

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Transitive Sets

Theorem

A set A is a transitive set iff $a \in A$ implies $a \subseteq A$.

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Theorem

A set A is a transitive set iff $a \in A$ implies $a \subseteq A$.

Proof (only if).

Let A be a transitive set and let $a \in A$. If $x \in a$ implies $x \in A$ because A is transitive. ∎

Proof (if).

Let $a \in A$ implies $a \subseteq A$. If $x \in a \land a \in A$ implies $x \in A$ because $a \subseteq A$. ∎

Transitive Sets

Theorem

A set A is a transitive set iff $A \subseteq \mathcal{P}(A)$.

Transitive Sets

On transitive sets

Let A be a set. Transitive sets can be defined using any of the followings equivalent affirmations:

$$x \in a \in A \text{ implies } x \in A,$$
$$\bigcup A \subseteq A,$$
$$a \in A \text{ implies } a \subseteq A,$$
$$A \subseteq \mathcal{P}(A).$$

Transitive Sets

Theorem (Enderton [1977, Theorem 4E])

If a is a transitive set, then

$$\bigcup \mathrm{S}(a) = a.$$

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Every natural number is a transitive set.

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If a is a transitive set, then

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Theorem (Enderton [1977, Theorem 4F])

Every natural number is a transitive set.

Theorem (Enderton [1977, Theorem 4G])

The set $\mathbf{N}$ is a transitive set.

Induction Principle for Natural Numbers

Induction principle for $\mathbf{N}$ (p. 42)

Let $P(x)$ be a property. Assume that

i) $P(0)$ holds and

ii) for all $n \in \mathbf{N}$, $P(n)$ implies $P(\mathrm{S}(n))$.

Then P holds for all natural numbers n.

Induction Principle for Natural Numbers

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Then P holds for all natural numbers n.

Proof.

> *'This is an immediate consequence of our definition of w. The assumptions i) and ii) simple say that the set $A = \{\, n \in \mathbf{N} \mid P(n) \,\}$ is inductive. $\mathbf{N} \subseteq A$ follows.' (p. 42)*

■

Induction Principle for Natural Numbers

Induction principle for $\mathbf{N}$ (equivalent version) (Enderton [1977, p. 69])

Any inductive subset of $\mathbf{N}$ coincides with $\mathbf{N}$.

Induction as Foundations

'Thus inductive definibility is a notion intermediate in strength between predicate and fully impredicative definability. It would be interesting to formulate a coherent conceptual framework that made induction the principal notion. There are suggestions of this in the literature, but the possibility has not yet been fully explored.' (Aczel 1977, p. 780)

Recursion on Natural Numbers

Recursion theorem on $\mathbf{N}$

For any set A, any $a \in A$, and any function $g : A \times \mathbf{N} \to A$, there exists a unique infinite sequence $f : \mathbf{N} \to A$ such that

(i) $f_0 = a$,

(ii) $f_{n+1} = g(f_n, n)$, for all $n \in \mathbf{N}$.

Operations and Structures

Definition

A **structure** is a pair (A, R) consisting of a set A and a binary relation R on A.

References

Aczel, Peter (1977). An Introduction to Inductive Definitions. In: Handbook of Mathematical Logic. Ed. by Barwise, Jon. Vol. 90. Studies in Logic and the Foundations of Mathematics. Elsevier. Chap. C.7. DOI: 10.1016/S0049-237X(08)71120-0 (cit. on p. 38).

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Hrbacek, Karel and Jech, Thomas [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on pp. 2, 8).