CM0832 Elements of Set Theory
Axioms and Operations

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Hrbacek and Jech (1978) 1999].

Extensionality Axiom

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If two sets have exactly the same members, then they are equal, that is,

$$\forall A \, \forall B \, [\forall x \, (x \in A \leftrightarrow x \in B) \rightarrow A = B].$$

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$$\forall A \forall B \, [\forall x \, (x \in A \leftrightarrow x \in B) \to A = B \,].$$

Question

Have we any set? No, we haven't.

Some Axioms for Building Sets

Empty (existence) axiom

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Pairing axiom

For any sets u and v, there is a set having as members just u and v, that is,

$$\forall a \, \forall b \, \exists C \, \forall x \, (x \in C \leftrightarrow x = a \vee x = b).$$

Some Axioms for Building Sets

Union axiom (first version)

For any sets a and b, there is a set whose members are those sets belonging either to a or to b (or both), that is,

$$\forall a \, \forall b \, \exists B \, \forall x \, (x \in B \leftrightarrow x \in a \vee x \in b).$$

Power set axiom

For any set a, there is a set whose members are exactly the subsets of a, that is,

$$\forall a \, \exists B \, \forall x \, (x \in B \leftrightarrow x \subseteq a),$$

where

$$u \subseteq v :\equiv \forall t \, (t \in u \rightarrow t \in v).$$

Some Axioms for Building Sets

Set abstraction operator

We added to the logical language of set theory the set abstraction operator $\{\, x \mid \varphi(x) \,\}$, where x is a variable and $\varphi(x)$ is a propositional function.

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Observation

We added the set abstraction operator for naming sets, but this operator can be eliminated (see, e.g. [Drake 1974, § 2.6] and [Potter 1990, § 1.1]).

Some Axioms for Building Sets

Definitions from the empty, pairing, union and power set axioms via set abstraction

Let a, b, u and v be sets, then we define

$$
\begin{aligned}
\emptyset &:= \{\, x \mid x \neq x \,\} && \text{(empty set)},\\
\{u, v\} &:= \{\, x \mid x = u \lor x = v \,\} && \text{(pair set)},\\
\{u\} &:= \{u, u\} && \text{(singleton set)},\\
a \cup b &:= \{\, x \mid x \in a \lor x \in b \,\} && \text{(union)},\\
\mathcal{P}(a) &:= \{\, x \mid x \subseteq a \,\} && \text{(power set)}.
\end{aligned}
$$

Some Axioms for Building Sets

Observation

Recall that our set of non-logical symbols is $\mathfrak{L} = \{\epsilon\}$. When we add some definitions, we formally are changing this set (e.g. $\mathfrak{L} = \{\epsilon, \emptyset, \cup\}$). See, e.g. [Kunen (2011) 2013, § I.2], [Kunen (1980) 1992, § I.8 and § I.13] and [Suppes (1960) 1972, § 2.1] for how to add valid definitions and how to handle the new sets of non-logical symbols created by these definitions.

Subset Axiom Scheme

Introduction

Whiteboard.

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Subset axiom scheme (axiom scheme of comprehension/separation)

For any propositional function $\varphi(x)$, not containing B, the following is an axiom:

$$\forall c \, \exists B \, \forall x \, (x \in B \leftrightarrow x \in c \land \varphi(x)).$$

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We stated an axiom scheme.

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Abstraction from the subset axiom scheme

$\{\, x \in c \mid \varphi(x) \,\}$ is the set of all $x \in c$ satisfying the property φ.

Subset Axiom Scheme

Observation

The propositional function φ can depend on other variables $t_1, \ldots, t_k$. In this case, we use $\varphi(x, t_1, \ldots, t_k)$ and we universally quantify on variables $t_1, \ldots, t_k$ when using the axiom scheme.

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Theorem (Enderton [1977, Theorem 2A])

There is no set to which every set belongs.

Proof

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Theorem (Enderton [1977, Theorem 2A])

There is no set to which every set belongs.

Proof

Whiteboard.

Exercise

Why does the subset axiom scheme avoid the Berry paradox?

Arbitrary Unions and Intersections

Union axiom (final version)

For any set A, there exists a set B whose elements are exactly the members of the members of A, that is,

$$\forall A\, \exists B\, \forall x\, [\, x \in B \leftrightarrow \exists b\, (x \in b \wedge b \in A)\,].$$

Arbitrary Unions and Intersections

Definition

Let A be a set. The **union** $\bigcup A$ of A is defined by

$$\bigcup A := \{\, x \mid \exists b\,(x \in b \land b \in A)\,\}.$$

Example (informal)

Let $A = \{\{2,4,6\},\{6,16,26\},\{0\}\}$, then

$$\bigcup A = \{0,2,4,6,16,26\}.$$

Example

$$\begin{aligned} a \cup b &= \bigcup\{a,b\}, \\ \bigcup\{a\} &= a, \\ \bigcup\emptyset &= \emptyset. \end{aligned}$$

Arbitrary Unions and Intersections

Theorem (Enderton [1977, Theorem 2B])

For any non-empty set A, there exists a unique set B such that for any x,

$$x \in B \quad \text{iff} \quad x \text{ belongs to every member of } A.$$

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Definition

Let A be a non-empty set. The **intersecction** $\bigcap A$ of A can be defined by

$$\bigcap A := \{\, x \mid \forall b\, (b \in A \to x \in b)\, \}, \text{ for } A \neq \emptyset.$$

Algebra of Sets

Exercise (Enderton [1977, Exercise 2.18])

Assume that A and B are subsets of S. List all of the different sets that can be made from these three by use of the binary operations $\cup$, $\cap$, and $-$.

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The Venn diagram shows four possible regions for shading, that is, we have 2^4 different sets given by

$\emptyset$, A, B, S, $A \cup B$, $A \cap B$, $A - B$, $B - A$, $A + B$, $S - A$, $S - B$, $S - (A \cup B)$, $S - (A \cap B)$, $S - (A - B)$, $S - (B - A)$ and $S - (A + B)$,

where the binary operation $+$ is the **symmetric difference** defined by

$$\begin{aligned} A + B &:= (A - B) \cup (B - A) \\ &= (A \cup B) - (A \cap B). \end{aligned}$$

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