CM0081 Automata and Formal Languages
§ 8.1 Problems That Computers Cannot Solve

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Preliminaries

Conventions

- The number and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook [Hopcroft, Motwani and Ullman 2007].
- The natural numbers include the zero, that is, $\mathbb{N} = \{0, 1, 2, \dots\}$.
- The power set of a set A, that is, the set of its subsets, is denoted by $\mathcal{P}\,A$.

Undecidable Problems

Recall

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Definition (informally)

A problem is **undecidable** iff no algorithm can solve it.

Question

Are there undecidable problems?

Undecidable Problems

Theorem

There are undecidable problems.

Proof (by cardinality)

Let Σ be an alphabet and let $\overline{\overline{S}}$ be the cardinality of the set S.

(i) Since Σ is an alphabet, then $\overline{\overline{\Sigma}} = n$, with $n \in \mathbb{Z}^+$, that is, the alphabet Σ is an enumerable set.

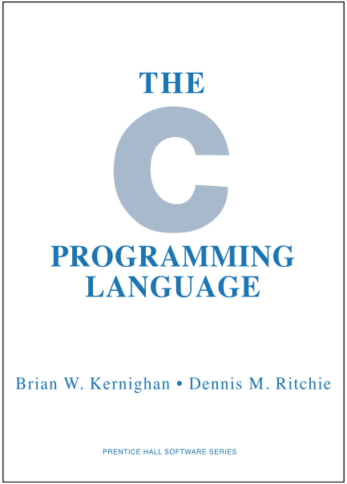
(ii) The set of all the strings over Σ, that is Σ^*, is an enumerable set (lexicographical order).

(iii) The set of all the languages over Σ, that is, the set $\{\, L \mid L \subseteq \Sigma^* \,\}$ is infinite not enumerable set.

(iv) The set $\{\, P \mid P$ is a program in a general programming language $\}$ is an enumerable set (lexicographical order).

Hence, there are more languages over Σ than programs. In consequence, there must be languages whose decision problems are undecidable. ■

The `hello-world` Problem: An Undecidable Problem



'The first program to write is the same for all languages: Print the words hello, world.' [1978, §1.1]

The `hello-world` Problem: An Undecidable Problem

Problem description

To determine whether a given program, with a given input, prints hello, world as the first 12 characters that it prints.

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To determine whether a given program, with a given input, prints hello, world as the first 12 characters that it prints.

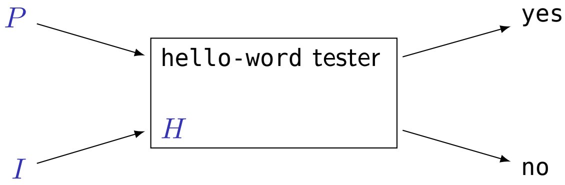
Theorem

The `hello-world` problem is an undecidable problem.

The hello-world Problem: An Undecidable Problem

Informal proof

1. We assume that the following program H exists:



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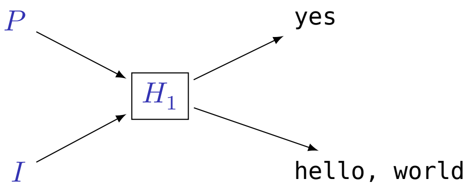
The `hello-world` Problem: An Undecidable Problem

2. From the program H build the program H_1 with the following behaviour:



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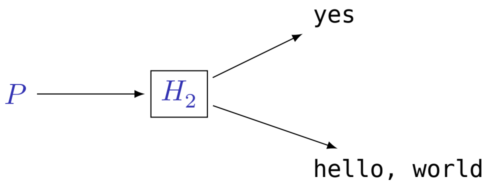
The `hello-world` Problem: An Undecidable Problem

3. To build the program H_2 by restricting H_1 to take only the input P (i.e. P is also the input of the program P):



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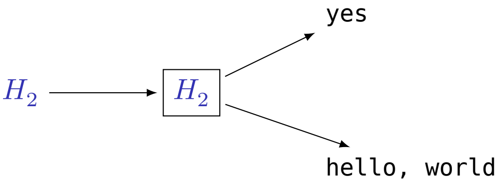
The hello-world Problem: An Undecidable Problem

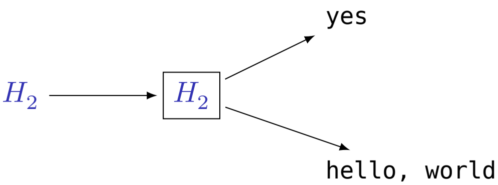
4. What does the program H_2 do when given itself as input?



Analysis on the whiteboard.

The `hello-world` Problem: An Undecidable Problem

4. What does the program H_2 do when given itself as input?



   Analysis on the whiteboard.

   Therefore the program H_2 can not exist.

5. Therefore the program H can not exists, i.e. the `hello-world` problem is undecidable. ■

Computability: Historical Remarks

1900 David Hilbert's problems
Wir müssen wissen — wir werden wissen!
(We must know — we will know!)

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1934-1937 Confluence of ideas

- Derivability from a system of equations (Kurt Gödel, Jacques Herbrand)
- Lambda calculus (Alonzo Church, Stephen C. Kleene, J. Barkley Rosser)
- Recursive functions (Kurt Gödel, Stephen C. Kleene)
- Post machines (Emil L. Post)
- Turing machines (Alan M. Turing)

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Computability: Historical Remarks

1936–40 The Church-Turing-Kleene thesis: A (number-theoretic) function is effectively calculable if and only if there is a Turing machine which computes it.

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- **1985** Deutsch's quantum Turing machines
- **1940–today** Many equivalents models of computation

Problems Reduction

How to prove that a problem is undecidable?

- Proof by contradiction (or constructively speaking, proof of negation [Bauer 2017])

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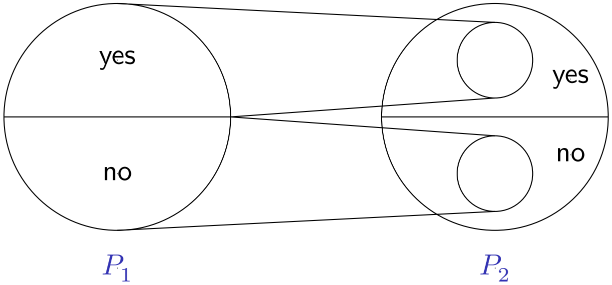
How to prove that a problem is undecidable?

- Proof by contradiction (or constructively speaking, proof of negation [Bauer 2017])
- Problem reduction

  Reduction from a problem P_1 to a problem P_2:



Problems Reduction

Theorem

If P_1 is undecidable and there is a reduction of P_1 to P_2, then P_2 is undecidable too.

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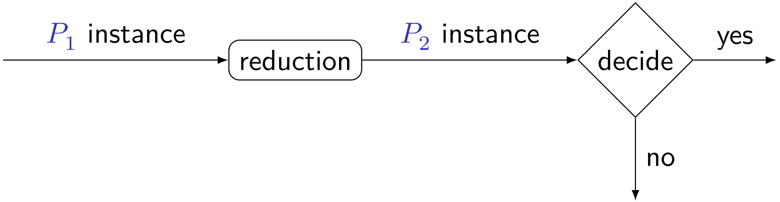
Proof by contradiction

Let's suppose P_2 decidable. The reduction from P_1 to P_2 implies that P_1 is decidable which is a contradiction. Therefore, P_2 is undecidable.



■

Problems Reduction

Example (the `calls-foo` problem)

Problem description: Does a program Q with an input z calls the function `foo`?

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Prove that the `calls-foo` problem is undecidable.

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Problems Reduction

Proof

Idea: Reduction of the `hello-world` problem to the `calls-foo` problem.

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That is, a program P with input y prints `hello, world` iff a program Q with input z calls the function `foo`.

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Reduction:

1. Program P_1: Rename `foo` in the program P.
2. Program P_2: Add the function `foo` to the program P_1.
3. Program P_3: Save the first 12 characters that prints the program P_2.
4. Program P_4: When the program P_3 executes an output statement if output is `hello, world` then calls the function `foo`.
5. $Q = P_4$ and $y = z$. ∎

Problems Reduction

Exercise 8.1.1.a (the halting problem)

Problem description: Given a program and an input, does the program eventually halt; i.e. does the program not loop forever on the input?

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Prove that the halting problem is undecidable.

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Problems Reduction

Proof (based on the solution in Hopcroft, Motwani and Ullman [2007])

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Idea: Reduction of the `hello-world` problem to the halting problem.

That is, a program P with input y prints `hello, world` iff a program Q with input z halts.

Reduction:

1. Program P_1: Add an infinite loop (e.g. `while(1);`) to the end of `main()`.
2. Program P_2: Save the first 12 characters that prints the program P_1.
3. Program P_3: When the program P_2 executes an output statement if output is `hello, world` then the program P_3 halts by going to the end of `main`.
4. $Q = P_3$ and $y = z$. ∎

Problems Reduction

Theorem

If P_1 is undecidable and there is a reduction of P_1 to P_2, then P_2 is undecidable too.

Be careful

In the above theorem the reduction is from P_1 to P_2, it is not from P_2 to P_1:

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$$P_2 \text{ decidable} \Rightarrow P_1 \text{ decidable}$$
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- From a reduction of P_2 to P_1:

$$P_1 \text{ decidable} \Rightarrow P_2 \text{ decidable} \qquad \text{(hypothesis false)}$$
$$P_2 \text{ undecidable} \Rightarrow P_1 \text{ undecidable} \qquad \text{(hypothesis unknown)}$$

References

Bauer, A. (2017). Five States of Accepting Constructive Mathematics. Bulletin of the American Mathematical Society 54.3, pp. 481–498. DOI: 10.1090/bull/1556 (cit. on pp. 20, 21).

Hopcroft, J. E., Motwani, R. and Ullman, J. D. [1979] (2007). Introduction to Automata Theory, Languages, and Computation. 3rd ed. Pearson Education (cit. on pp. 2, 31–33).