CM0081 Automata and Formal Languages
§ 2.5 Finite Automata with Epsilon-Transitions

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Preliminaries

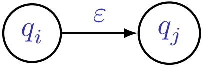
Conventions

- The number and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook [Hopcroft, Motwani and Ullman 2007].
- The natural numbers include the zero, that is, $\mathbb{N} = \{0, 1, 2, \dots\}$.
- The power set of a set A, that is, the set of its subsets, is denoted by $\mathcal{P}\,A$.

Finite Automata with Epsilon-Transitions

Introduction

- An automaton with ε-transitions is a NFA where transition with the empty string are allowed.



Finite Automata with Epsilon-Transitions

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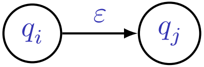


- The ε-transitions do not increase the computational power (or expressive power) of finite automata.

Finite Automata with Epsilon-Transitions

Introduction

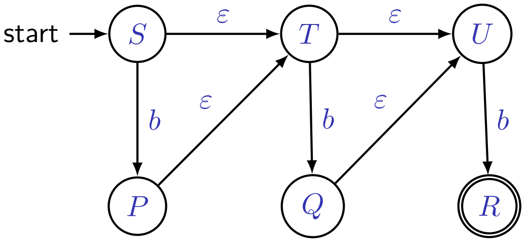
- An automaton with ε-transitions is a NFA where transition with the empty string are allowed.

$$q_i \xrightarrow{\varepsilon} q_j$$

- The ε-transitions do not increase the computational power (or expressive power) of finite automata.
- The ε-transitions facilitate the design of the automata.

Finite Automata with Epsilon-Transitions

Example ([Kozen 2012])



The automaton accepts the language $\{b, bb, bbb\}$.

Finite Automata with Epsilon-Transitions

Example

A finite automaton with epsilon-transitions that accepts decimal numbers consisting of:

(i) an optional $+$ or $-$ sign,

(ii) a string of digits,

(iii) a decimal point and

(iv) another string of digits. Either this string of digits, or the string (ii) can be empty, but at least one of the two strings must be non-empty.

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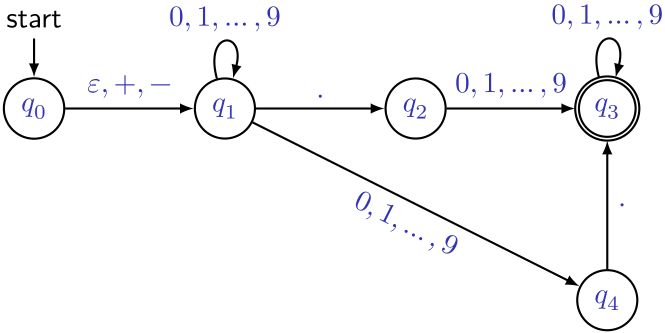
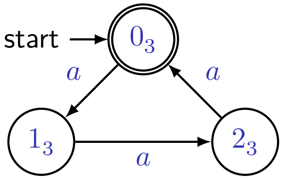
Finite Automata with Epsilon-Transitions

Example (continuation)

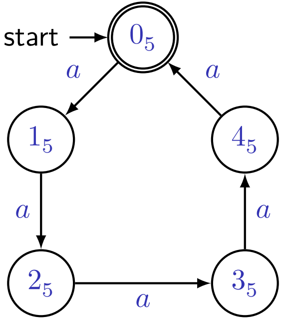


Finite Automata with Epsilon-Transitions

Example



$\{ w \in \{a\}^* \mid |w|$ is divisible by $3 \}$



$\{ w \in \{a\}^* \mid |w|$ is divisible by $5 \}$

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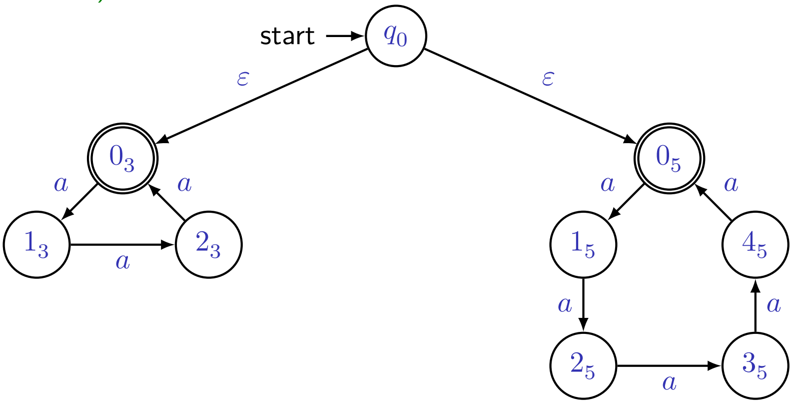
Finite Automata with Epsilon-Transitions

Example (continuation)



$$\{\, w \in \{a\}^* \mid |w| \text{ is divisible by } 3 \text{ or by } 5 \,\}$$

Finite Automata with Epsilon-Transitions

Definition

A **finite automaton with epsilon-transitions** (ε-NFA) is a 5-tuple

$$(Q, \Sigma, \delta, q_0, F),$$

where

(i) Q is the finite set of states,

(ii) Σ is the alphabet of input symbols,

(iii) $\delta : Q \times \Sigma\{\varepsilon\} \to \mathcal{P}\, Q$ is the transition function,

(iv) $q_0 \in Q$ is the start state,

(v) $F \subseteq Q$ is the set of accepting (or final) states.

Epsilon-Closures

Definition

The **ε-closure** of a state q, denoted by $\mathrm{eclose} : Q \to \mathcal{P}\,Q$, are all states reachable from q by a sequence $\varepsilon \cdots \varepsilon$. We recursively define eclose by:

$$\frac{}{q \in \mathrm{eclose}(q)} \text{ if } q \in Q \qquad\qquad \frac{p \in \mathrm{eclose}(q) \qquad \delta(p,\varepsilon) = r}{r \in \mathrm{eclose}(q)}$$

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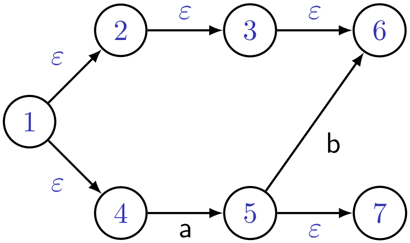
Example



Epsilon-Closures

Definition

Let S be a set of states. The ε**-closure** of S is defined by

$$\mathrm{eclose}(S) := \bigcup_{q \in S} \mathrm{eclose}(q).$$

Extension of the Transition Function for ε-NFAs

Definition

Let $E = (Q, \Sigma, \delta, q_0, F)$ be an ε-NFA. The **extension of the transition function**, denoted by $\hat{\delta}$, is recursively defined by

$$
\begin{aligned}
\hat{\delta} : Q \times \Sigma^* &\to \mathcal{P}\, Q \\
\hat{\delta}(q, \varepsilon) &= \mathrm{eclose}(q), \\
\hat{\delta}(q, xa) &= \mathrm{eclose}(\{r_1, r_2, \dots, r_m\}),
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\delta}(q, x) &= \{p_1, p_2, \dots, p_k\}, \\
\bigcup_{i=1}^{k} \delta(p_i, a) &= \{r_1, r_2, \dots, r_m\}.
\end{aligned}
$$

Languages Accepted by ε-NFAs

Recall

Let $D = (Q, \Sigma, \delta, q_0, F)$ and $N = (Q, \Sigma, \delta, q_0, F)$ be a DFA and a NFA, respectively. Recall that the languages accepted by D and N were defined by

$$\mathrm{L}(D) := \left\{ w \in \Sigma^* \;\middle|\; \hat{\delta}(q_0, w) \in F \right\},$$
$$\mathrm{L}(N) := \left\{ w \in \Sigma^* \;\middle|\; \hat{\delta}(q_0, w) \cap F \neq \emptyset \right\}.$$

Languages Accepted by ε-NFAs

Definitions

Let $E = (Q, \Sigma, \delta, q_0, F)$ be a ε-NFA and let $w \in \Sigma^*$ be a string.

(i) The string w is **accepted** by E iff $\hat{\delta}(q_0, w) \cap F \neq \emptyset$.

Languages Accepted by ε-NFAs

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(i) The string w is **accepted** by E iff $\hat{\delta}(q_0, w) \cap F \neq \emptyset$.

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Languages Accepted by ε-NFAs

Definitions

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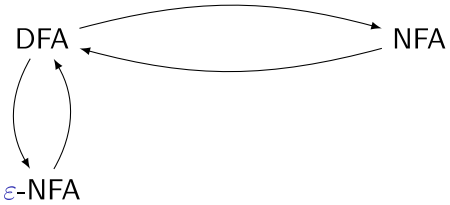
(ii) The string w is **rejected** by E iff $\hat{\delta}(q_0, w) \cap F = \emptyset$.

(iii) The **language accepted** by E, denoted $\mathrm{L}(E)$, is the set of strings accepted by E, that is,

$$\mathrm{L}(E) := \left\{ w \in \Sigma^* \;\middle|\; \hat{\delta}(q_0, w) \cap F \neq \emptyset \right\}.$$

Equivalence of DFAs, NFAs and ε-NFAs



Regular Languages

Definition

A language L is **regular** iff exists a finite automaton A (DFA, NFA or ε-NFA) such that $L = \mathrm{L}(A)$.

Exercise 2.5.3.a

Let L be the set of strings consisting of zero or more a's followed by zero or more b's, followed by zero or more c's. Prove that L is a regular language.

References

- Hopcroft, J. E., Motwani, R. and Ullman, J. D. [1979] (2007). Introduction to Automata Theory, Languages, and Computation. 3rd ed. Pearson Education (cit. on p. 2).
- Kozen, D. C. [1997] (2012). Automata and Computability. Third printing. Undergraduate Texts in Computer Science. Springer. DOI: 10.1007/978-1-4612-1844-9 (cit. on p. 6).