CM0081 Automata and Formal Languages
§ 9.1 A Language That Is Not Recursively Enumerable

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Preliminaries

Conventions

- The number and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook [Hopcroft, Motwani and Ullman 2007].
- The natural numbers include the zero, that is, $\mathbb{N} = \{0, 1, 2, \dots\}$.
- The power set of a set A, that is, the set of its subsets, is denoted by $\mathcal{P}\, A$.

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Definition

A language L is **undecidable** iff L is not recursive.

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- Equivalent formalization to Turing-machine computability based on recursive functions.
- A function is recursive if only if it is Turing-machine computable (see, e.g. [Boolos, Burges and Jeffrey 2007], [Hermes 1969] or [Kleene 1974]).
- Recursive problem: *‘it is sufficiently simple that I can write a recursive function to solve it, and the function always finishes.’ [p. 385]*

Codification of Turing Machines

Convention

The Turing machine M is of the form:

$$M = (\{q_1, \dots, q_n\}, \{0, 1\}, \{X_1, X_2, X_3, \dots, X_m\}, \delta, q_1, B, \{q_2\}),$$

where $X_1 = 0$, $X_2 = 1$ and $X_3 = B$. Moreover, $D_1 = L$ and $D_2 = R$.

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Codification of an instruction

The instruction $\delta(q_i, X_j) = (q_k, X_l, D_m)$ is codified by

$$0^i 1 0^j 1 0^k 1 0^l 1 0^m.$$

Codification of Turing Machines

Codification of a Turing machine

Let $C_1, C_2, \dots, C_p$ be the codifications of the instructions of a Turing machine M. The codification of M is defined by

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Observation

Note that there are other possible codes for M.

Codification of Turing Machines

Enumeration of the binary strings

We ordered the binary strings by [length-]lexicographical order (strings are ordered by length, and strings of equal length are ordered lexicographically).

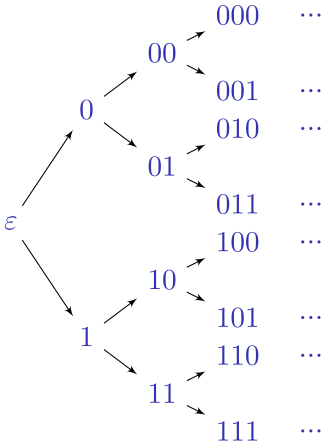
(continued on next slide)

Codification of Turing Machines

Enumeration of the binary strings (continuation)

If w is a binary string, we call w the i-th string where $1w$ is the binary integer i. We refer to the i-th string as w_i.

$$
\begin{aligned}
\varepsilon &\to 1_b &&\to 1,\\
0 &\to 10_b &&\to 2,\\
1 &\to 11_b &&\to 3,\\
00 &\to 100_b &&\to 4,\\
01 &\to 101_b &&\to 5,\\
10 &\to 110_b &&\to 6,\\
&\vdots
\end{aligned}
$$



Codification of Turing Machines

i-th Turing machine

Given a Turing machine M with code w_i, we can now associate a natural number to it: M is the i-th Turing machine, referred to as M_i.

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Convention

If w_i is not a valid Turing machine code, we shall take M_i to be the Turing machine with one state and no transitions, that is,

$$\mathrm{L}(M_i) = \emptyset.$$

Cantor's Diagonalisation Proof

Theorem

The open interval $(0, 1)$ is an uncountable (non-enumerable) set.

(continued on next slide)

Cantor's Diagonalisation Proof

Proof.

Let's suppose $(0, 1)$ is (infinite) countable.

$$
\begin{aligned}
r_1 &= 0.d_{11}d_{12}d_{13}d_{14}\ldots \\
r_2 &= 0.d_{21}d_{22}d_{23}d_{24}\ldots \\
r_3 &= 0.d_{31}d_{32}d_{33}d_{34}\ldots \\
&\vdots
\end{aligned}
$$

Let $r = 0.d_1d_2d_3\ldots \in (0, 1)$, where

$$
d_i = \begin{cases} 4, & \text{if } d_{ii} \neq 4; \\ 5, & \text{if } d_{ii} = 4. \end{cases}
$$

The number r does not belong to the above enumeration. Therefore the interval $(0, 1)$ is an uncountable set. ■

The Diagonalization Language

Definition

Let $\Sigma = \{0, 1\}$. The **diagonalization language** is defined by

$$\mathrm{L_d} := \{\, w_i \in \Sigma^* \mid w_i \notin \mathrm{L}(M_i) \,\}.$$

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		w_j 1	2	3	4	...
M_i	1	0	1	1	0	...
	2	1	1	0	1	...
	3	0	1	1	0	...
	4	1	1	0	0	...
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$$a_{ij} = \begin{cases} 1, & \text{if } w_j \in \mathrm{L}(M_i); \\ 0, & \text{if } w_j \notin \mathrm{L}(M_i). \end{cases}$$

Language $\mathrm{L}(M_i)$'s vector: i-th row

$\mathrm{L_d}$: Complement of the diagonal

Is it possible that $\mathrm{L_d}$ be in a row?

The Diagonalization Language

Theorem 9.2

The language $\mathrm{L_d}$ is not recursively enumerable.

Proof by contradiction (proof of negation)

Whiteboard.

References

- Boolos, G. S., Burges, J. P. and Jeffrey, R. C. [1974] (2007). Computability and Logic. 5th ed. Cambridge University Press (cit. on pp. 6–9).
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