CM0246 Discrete Structures
Representing Relations

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Matrices

Definition

A **matrix** is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix.

$$\boldsymbol{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Matrices

Definition

Let $\boldsymbol{A}_{m\times k}$ and $\boldsymbol{B}_{k\times n}$ be two matrices. The product $\boldsymbol{AB}$ is the matrix $m\times n$ defined by

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{a_{i1}} & \boldsymbol{a_{i2}} & \cdots & \boldsymbol{a_{ik}} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & \boldsymbol{b_{1j}} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & \boldsymbol{b_{2j}} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{k1} & b_{k2} & \cdots & \boldsymbol{b_{kj}} & \cdots & b_{kn} \end{bmatrix} =$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \boldsymbol{c_{ij}} & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}.$$

Matrices

Definition

Let $\boldsymbol{A} = [a_{ij}]$ be a $m \times n$ matrix. The **transpose** of $\boldsymbol{A}$, denoted by $\boldsymbol{A}^t$, is the $n \times m$ matrix obtained by interchanging the rows and columns of $\boldsymbol{A}$.

Matrices

Boolean operations

$$b_1 \wedge b_2 = \begin{cases} 1, & \text{if } b_1 = b_2 = 1; \\ 0, & \text{otherwise,} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1, & \text{if } b_1 = 1 \text{ or } b_2 = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Matrices

Definition

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$ Booleans matrices. The **join** (*unión*)/**meet** (*intersección*) of $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted $\boldsymbol{A} \vee \boldsymbol{B}/\boldsymbol{A} \wedge \boldsymbol{B}$, is the Boolean matrix with (i, j)-th entry $a_{ij} \vee b_{ij}/a_{ij} \wedge b_{ij}$.

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Example

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

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$$\boldsymbol{A} \wedge \boldsymbol{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Matrices

Definition

Let $\boldsymbol{A}_{m\times k}$ and $\boldsymbol{B}_{k\times n}$ be two Boolean matrices. The **Boolean product** of $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted $\boldsymbol{A} \odot \boldsymbol{B}$, is the Boolean matrix $m \times n$ with (i, j)-th entry

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

Matrices

Example (product of Boolean matrices)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

Matrices

Example (product of Boolean matrices)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$\begin{aligned} \boldsymbol{A} \odot \boldsymbol{B} &= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix} \\ &= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \end{aligned}$$

Representing Relations Using Boolean Matrices

Representing of relations using Boolean matrices

Let R be a relation from a A to B. The relation R can be represented by the Boolean matrix $\boldsymbol{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R; \\ 0, & \text{otherwise.} \end{cases}$$

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Example

Whiteboard.

Representing Relations Using Boolean Matrices

Properties of relations from their Boolean matrix representation

See slides § 8.3 for the 6th ed. of Rosen's textbook.

Representing Relations Using Boolean Matrices

Boolean matrix representing of the combination of relations

$$
\begin{aligned}
\boldsymbol{M}_{R_1 \cup R_2} &= \boldsymbol{M}_{R_1} \vee \boldsymbol{M}_{R_2}, \\
\boldsymbol{M}_{R_1 \cap R_2} &= \boldsymbol{M}_{R_1} \wedge \boldsymbol{M}_{R_2}, \\
\boldsymbol{M}_{S \circ R} &= \boldsymbol{M}_{R} \odot \boldsymbol{M}_{S}.
\end{aligned}
$$

Representing Relations Using Boolean Matrices

Example

$$\boldsymbol{M}_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad \boldsymbol{M}_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array},$$

$$\boldsymbol{M}_{S \circ R} = \boldsymbol{M}_R \odot \boldsymbol{M}_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}.$$

Representing Relations Using Digraphs

Definition

A **digraph** (or **directed graph**) consists of a set V of vertices together with a set E of ordered pairs of elements of V called edges.

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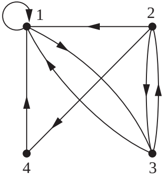
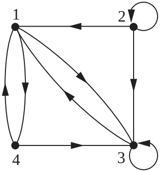
Representing relations using digraphs

The relation R on a set A is represented by a digraph where $V = A$ and (a, b) is an edge if $(a, b) \in R$.

Representing Relations Using Digraphs†

Example

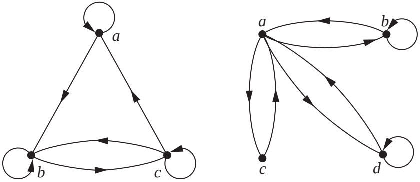




†Figure source: (Rosen 2012, § 9.3, Figs. 4 and 5).

Representing Relations Using Digraphs†

Properties of relations



† Figure source: (Rosen 2012, § 9.3, Fig. 6).

References

Rosen, K. H. (2012). Discrete Mathematics and Its Applications. 7th ed. McGraw-Hill (cit. on pp. 20, 21).