CM0246 Discrete Structures
Representing Graphs and Graph Isomorphism

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook (Rosen 2004).

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Definition 1

Two vertices u and v in an undirected graph G are called **adjacent** in G if $\{u, v\}$ is an edge of G. If $e = \{u, v\}$, the edge e is called **incident** with the vertices u and v.

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Degrees of the Vertices

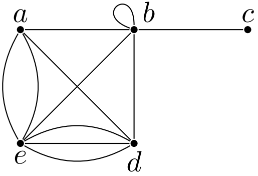
Definition

The **degree of a vertex** v in an undirected graph, denoted $\delta(v)$, is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.

Degrees of the Vertices

Exercise

Find the degree of each vertex in the following graph:



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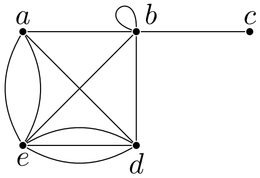

Solution

$\delta(a) = 4$, $\delta(b) = \delta(e) = 6$, $\delta(c) = 1$ and $\delta(d) = 5$.

Vertex Degrees

Theorem 1 (the handshaking theorem, p. 511)

Let $G = (V, E)$ be an undirected graph with e edges, then

$$2e = \sum_{v \in V} \delta(v).$$

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Proof.

Each edge contributes two to the sum of the degrees of the vertices because an edge is incident with exactly two (possibly equal) vertices. This means that the sum of the degrees of the vertices is twice the number of edges. ■

Representing Graphs

- Adjacency matrices
- Incidence matrices

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The **adjacency matrix** $\boldsymbol{A}_G = [a_{ij}]$ of G is a $n \times n$ matrix, where

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \text{ is an edge of } G; \\ 0, & \text{otherwise.} \end{cases}$$

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The **incidence matrix** $M_G = [m_{ij}]$ of G is a $n \times m$ Boolean matrix, where

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Isomorphism of Graphs

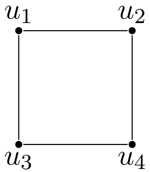
Definition

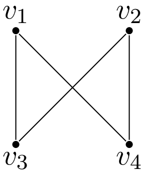
The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there exists a bijective function f from V_1 to V_2 with the property that u and v are adjacent in G_1, if and only if, $f(u)$ and $f(v)$ are adjacent in G_2, for all u and v in V_1.

Isomorphism of Graphs

Example

The following simple graphs are isomorphic.





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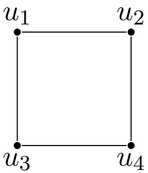

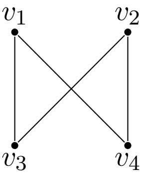

The bijective function f preserves adjacency.

$$f : \{u_1, u_2, u_3, u_4\} \to \{v_1, v_2, v_3, v_4\}$$

$f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$ and $f(u_4) = v_2$.

Isomorphism of Graphs

Remark

Determining whether two simple graphs are isomorphic is often difficult because if $|A| = |B| = n$ then

$$|\{ f : A \to B \mid f \text{ is a bijection} \}| = n!.$$

Isomorphism of Graphs

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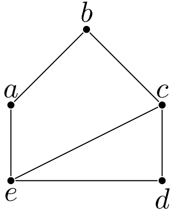
We can prove that two graphs are not isomorphic if we can find a graph invariant property that only one of the two graphs has.

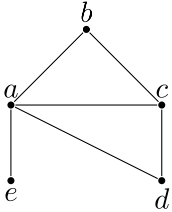
Isomorphism of Graphs

Example

Are the following graphs isomorphic?



G



H

Isomorphism of Graphs

Example

Are the following graphs isomorphic?

Solution

No. The graph H has a vertice of degree 1 but the graph G have no vertices of degree 1.

Isomorphism of Graphs

Definition

The **complementary graph** $\overline{G}$ of a simple graph G has the same vertices as G. Two (different) vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Example

Whiteboard.

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Problem 50 (p. 529)

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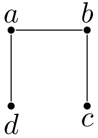

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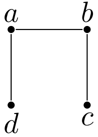
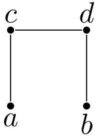


Yes! The complementary graph is given by the figure.



The isomorphism is $f(a) = c$, $f(b) = d$, $f(c) = b$ and $f(d) = a$.

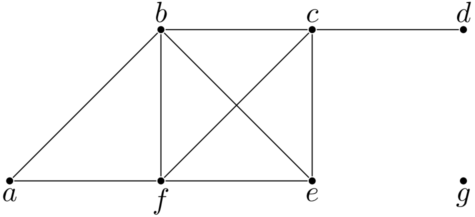
Isomorphism of Graphs

Definition

The **degree sequence** of a graph is the sequence of the degrees of the vertices of the graph in non-increasing order.

Example

For the graph in the figure, the degree sequence is $4, 4, 4, 3, 2, 1, 0$.



Isomorphism of Graphs

Problem 69 (p. 530)

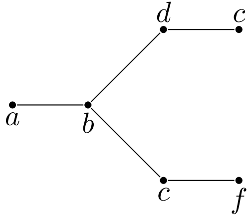
A counter-example for a purported isomorphism test is a pair of non-isomorphic graphs that the test fails to show that they are not isomorphic.

Find a counter-example for the test that checks the degree sequence in two graphs to make sure they agree.

Isomorphism of Graphs

Solution

The degree sequence of both graphs is $3, 2, 2, 1, 1, 1$ but they are not isomorphic. In graph G, the vertice b has degree 3 and it is adjacent to two vertices of degree 2 and one vertice of degree 1. The graph H has no vertice with these properties.



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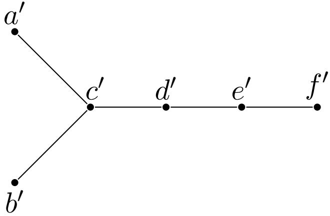

H

Isomorphism of Graphs

Comparison of several time complexity functions

$f(n)$	10	50	100
$\log n$	2.3 sec	3.9 sec	4.6 sec
n	10 sec	50 sec	1.7 min
n^2	1.7 min	41.7 min	2.8 h
2^n	17.1 min	358.001 c	4×10^{20} c
3^n	16.4 h	2.3×10^{14} c	1.6×10^{38} c
$n!$	42 d	9.7×10^{54} c	3×10^{148} c

Isomorphism of Graphs

Algorithms for graph isomorphism

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vertices	10	100	1000	10000
$2^{\sqrt{n \log n}}$	27.8 sec	33.4 d	3.3×10^{15} c	7.3×10^{81} c

References

Johnson, D. S. (2005). The NP-Completeness Column. ACM Transactions on Algorithms 1.1, pp. 160–176. DOI: 10.1145/1077464.1077476 (cit. on pp. 61, 62).

Rosen, K. H. (2004). *Matemática Discreta y sus Aplicaciones*. 5th ed. Translated by José Manuel Pérez Morales and others. McGraw-Hill (cit. on p. 2).