CM0246 Discrete Structures
Relations and Their Properties

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook (Rosen 2004).

Relations

Recall the definition of Cartesian product

Let A and B be sets. The Cartesian product of A and B is

$$A \times B = \{\, (a,b) \mid a \in A \wedge b \in B \,\}\,.$$

Example

Let $A = \{a,b\}$ and $B = \{1,2\}$. Then

$$A \times B = \{(a,1),(a,2),(b,1),(b,2)\}.$$

Relations

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Example

See whiteboard.

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See whiteboard.

Notation

We shall use $(a, b) \in R$ and $a \, R \, b$.

The slides for the 6th ed. of Rosen's textbook use $\langle a, b \rangle$.

Relations

Relations and functions

The functions are relations with additional constraints.

Relations

Definition

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Example

Some relations on $\mathbb{Z}$:

$$R_1 = \{ (a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a \leq b \},$$
$$R_2 = \{ (a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a > b \},$$
$$R_3 = \{ (a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a = b \vee a = -b \},$$
$$R_4 = \{ (a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a = b \},$$
$$R_5 = \{ (a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a = b + 1 \},$$
$$R_6 = \{ (a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a + b \leq 3 \}.$$

Properties of Relations

Definition

Let R be a relation on a set A. The relation R is

reflexive iff $\forall x(xRx)$,

symmetric iff $\forall x \forall y(xRy \rightarrow yRx)$,

antisymmetric iff $\forall x \forall y((xRy \wedge yRx) \rightarrow x = y)$ and

transitive iff $\forall x \forall y \forall z((xRy \wedge yRz) \rightarrow xRz)$.

Properties of Relations

Example

See slides § 8.1, p. 5 for the 6th ed. of Rosen's textbook.

Combing Relations

See slides § 8.1, pp. 6-8 for the 6th ed. of Rosen's textbook.

Combing Relations

Definition

Let R be a relation from A to B and let S be a relation from B to C.

The **composition** of S with R, denoted $S \circ R$, is the relation from A to C where if $(a, b) \in R$ and $(b, c) \in S$ then $(a, c) \in S \circ R$.

Combing Relations

Example (composition of relations)

Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$ and $C = \{0, 1, 2\}$.

Let R and S be the relations from A to B and B to C, respectively, given by

$$R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\},$$
$$S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\},$$

then $S \circ R$ is the relation from A to C, given by

$$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}.$$

Combing Relations

Problem 31 (p. 448)

Let R be the relation on the set of people consisting of pairs (a, b), where a is a parent of b. Let S be the relation on the set of people consisting of pairs (a, b), where a and b are siblings (brothers or sisters).

Combing Relations

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What are $S \circ R$ and $R \circ S$?

Combing Relations

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What are $S \circ R$ and $R \circ S$?

- $(a, b) \in S \circ R$ if exists c such that $(a, c) \in R$ (a is parent of c) and $(c, b) \in S$ (c is sibling of b), that is

$$S \circ R = \{ (a, b) \mid a \text{ is a parent of } b \text{ and } b \text{ has a sibling} \} .$$

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- $(a, b) \in R \circ S$ if exists c such $(a, c) \in S$ (a is sibling of c) and $(c, b) \in R$ (c is parent of b), that is

$$R \circ S = \{ (a, b) \mid a \text{ is an aunt or uncle of } b \} .$$

Combing Relations

Definition

Let R be a relation on the set A. The **powers** R^n, for $n \in \mathbb{Z}^+$ are defined recursively by

$$R^1 = R,$$
$$R^{n+1} = R^n \circ R.$$

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Example

See slides § 8.1, pp. 9-10 for the 6th ed. of Rosen's textbook.

Combing Relations

Theorem 1 (p. 446)

Let R be a relation on a set A. The relation R is transitive iff $R^n \subseteq R$ for $n \in \mathbb{Z}^+$.

Proved on next slides

Combing Relations

Proof of $\Rightarrow$ (if R is transitive implies $R^n \subseteq R$ for $n \in \mathbb{Z}^+$).

By induction on $n \in \mathbb{Z}^+$.

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1. $P(n)$: if R is transitive implies $R^n \subseteq R$.
2. Basis step $P(1)$: $R^1 = R \subseteq R$

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2. Basis step $P(1)$: $R^1 = R \subseteq R$
3. Inductive step:
   Inductive hypothesis $P(k)$: if R is transitive implies $R^k \subseteq R$

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   Let's prove $P(k+1)$:

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   3.1 Let $(a, b) \in R^{k+1}$.

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   Inductive hypothesis $P(k)$: if R is transitive implies $R^k \subseteq R$
   Let's prove $P(k+1)$:
   3.1 Let $(a,b) \in R^{k+1}$.
   3.2 Exists $x \in A$ such that $(a,x) \in R^k$ and $(x,b) \in R$ (definition of $\circ$).

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   3.3 $(a, x) \in R$ (IH).

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   Let's prove $P(k+1)$:
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   3.2 Exists $x \in A$ such that $(a, x) \in R^k$ and $(x, b) \in R$ (definition of $\circ$).
   3.3 $(a, x) \in R$ (IH).
   3.4 If $(a, x) \in R$ and $(x, b) \in R$ then $(a, b) \in R$ (R is transitive).

Combing Relations

Proof of $\Rightarrow$ (if R is transitive implies $R^n \subseteq R$ for $n \in \mathbb{Z}^+$).

By induction on $n \in \mathbb{Z}^+$.

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3. Inductive step:
   Inductive hypothesis $P(k)$: if R is transitive implies $R^k \subseteq R$
   Let's prove $P(k+1)$:
   3.1 Let $(a, b) \in R^{k+1}$.
   3.2 Exists $x \in A$ such that $(a, x) \in R^k$ and $(x, b) \in R$ (definition of $\circ$).
   3.3 $(a, x) \in R$ (IH).
   3.4 If $(a, x) \in R$ and $(x, b) \in R$ then $(a, b) \in R$ (R is transitive).
   3.5 $R^{k+1} \subseteq R$. ∎

Continued on next slide

Combing Relations

Proof of $\Leftarrow$ (if $R^n \subseteq R$ for $n \in \mathbb{Z}^+$ implies R is transitive).

1 Suppose that $(a, b) \in R$ and $(b, c) \in R$.

2 $(a, c) \in R^2$. (def. of R^2)

3 $(a, c) \in R$. ($R^2 \subseteq R$)

4 Therefore, R is transitive. ■

Inverse and Complementary Relations

Definition

Let R be a relation from A to B. The **inverse relation** from B to A, denoted by R^{-1}, is the set of ordered pairs

$$R^{-1} = \{ (b,a) \in B \times A \mid (a,b) \in R \} .$$

Inverse and Complementary Relations

Definition

Let R be a relation from A to B. The **complementary relation** from A to B, denoted by $\overline{R}$, is the set of ordered pairs

$$\overline{R} = \{ (a,b) \in A \times B \mid (a,b) \notin R \}.$$

References

Rosen, K. H. (2004). *Matemática Discreta y sus Aplicaciones*. 5th ed. Translated by José Manuel Pérez Morales and others. McGraw-Hill (cit. on p. 2).