CM0246 Discrete Structures
Recursive Definitions and Structural Induction

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook (Rosen 2004).

Introduction

Subjects

- Recursively defined functions from the natural numbers

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- Recursively defined functions from the natural numbers
- Inductively defined sets and structures

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Terminological note

We talk about 'recursively' defined functions and 'inductively' defined sets/structures. Both terms are used interchangeably in the textbook.

Introduction

Recursion

'Sometimes it is difficult to define an object explicitly. However, it may be easy to define this object in terms of itself. This process is called **recursion**.' (Rosen 2012, 7th ed. p. 344)

Recursively Defined Functions from the Natural Numbers

Recursively defined functions from the natural numbers

Let A be a set. A function $f : \mathbb{N} \to A$ is recursively defined if:

- Basis step: We define the function on $f(0)$.

Recursively Defined Functions from the Natural Numbers

Recursively defined functions from the natural numbers

Let A be a set. A function $f : \mathbb{N} \to A$ is recursively defined if:

- Basis step: We define the function on $f(0)$.
- Recursive step: We give a rule for $f(n)$ from the value of f on smaller natural numbers.

Recursively Defined Functions from the Natural Numbers

Example

$$f : \mathbb{N} \to \mathbb{N}$$
$$f(0) = 3,$$
$$f(n+1) = 2f(n) + 3.$$

Recursively Defined Functions from the Natural Numbers

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Example (the factorial function)

$$! : \mathbb{N} \to \mathbb{N}$$
$$0! = 1,$$
$$(n+1)! = (n+1)n!.$$

Recursively Defined Functions from the Natural Numbers

Example

Give a recursive definition of $\sum_{k=0}^{n} a_k$.

Recursively Defined Functions from the Natural Numbers

Example

Give a recursive definition of $\sum_{k=0}^{n} a_k$.

Solution

$$\sum_{k=0}^{0} a_k = a_0,$$
$$\sum_{k=0}^{n+1} a_k = \left(\sum_{k=0}^{n} a_k\right) + a_{n+1}.$$

Recursively Defined Functions from the Natural Numbers

Definition

Let A be a set. A function $f : \mathbb{N} \to A$ is **well defined** if for every natural number, the value of the function at this number is determined in an unambiguous way.

Recursively Defined Functions from the Natural Numbers

Problem 56 (p. 254)

Use mathematical induction to prove that a function f defined by specifying $f(0)$ and a rule for obtaining $f(n+1)$ from $f(n)$ is well defined.

Recursively Defined Functions from the Natural Numbers

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Scheme

The function $f : \mathbb{N} \to A$ is defined by

$$f(0) = c \in A,$$
$$f(n+1) = e[f(n)].$$

Recursively Defined Functions from the Natural Numbers

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Scheme

The function $f : \mathbb{N} \to A$ is defined by

$$f(0) = c \in A,$$
$$f(n+1) = e[f(n)].$$

Proof

Proof by mathematical induction (see whiteboard).

Recursively Defined Functions from the Natural Numbers

Example (Fibonacci numbers)

$$
\begin{aligned}
f &: \mathbb{N} \to \mathbb{N} \\
f(0) &= 0, \\
f(1) &= 1, \\
f(n+2) &= f(n-1) + f(n-2).
\end{aligned}
$$

Subjects

- Recursively defined functions from the natural numbers
- Inductively defined sets and structures
- Recursively defined functions from inductively defined sets and structures
- Structural induction

Inductively Defined Sets and Structures

Example (the natural numbers)

See whiteboard.

Inductively Defined Sets and Structures

Example (the natural numbers)

See whiteboard.

Definition

An **alphabet** is a finite, non-empty set of symbols.

Example

$$\Sigma_1 = \{0, 1\},$$
$$\Sigma_2 = \{a, b, \ldots, z\},$$
$$\Sigma_3 = \{\, x \mid x \text{ is an Unicode codepoint} \,\}.$$

Inductively Defined Sets and Structures

Definition

A **string** is a finite sequence of symbols of an alphabet. The empty string is denoted λ.

Example

See whiteboard.

Inductively Defined Sets and Structures

Definition

A **string** is a finite sequence of symbols of an alphabet. The empty string is denoted λ.

Example

See whiteboard.

Example (inductive definition of strings)

See whiteboard.

Inductively Defined Sets and Structures

Definition

A **string** is a finite sequence of symbols of an alphabet. The empty string is denoted λ.

Example

See whiteboard.

Example (inductive definition of strings)

See whiteboard.

Remark: See the correction to the textbook definition in course's website.

Inductively Defined Sets and Structures

Example (well-formed propositional logic formulae)

- Basis step: T, F and p (propositional variable) are well-formed formulae.
- Inductive step: If E and F are well-formed formulae, then $(\neg E)$, and $(E * F)$ with $* \in \{\wedge, \vee, \rightarrow\}$, are well-formed formulae.

Inductively Defined Sets and Structures

Inductively defined sets and structures

A set/structure is inductively defined if:

- Basis step: We define an initial collection of elements in the set/structure.

Inductively Defined Sets and Structures

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Inductively Defined Sets and Structures

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A set/structure is inductively defined if:

- Basis step: We define an initial collection of elements in the set/structure.
- Inductive step: We give rules for forming new elements in the set/structure from those already known to be in the set/structure.
- Exclusion rule: We specific that all the elements in the set/structure are those elements specified in the basis step or generated by applications of the inductive step.

Inductively Defined Sets and Structures

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A set/structure is inductively defined if:

- Basis step: We define an initial collection of elements in the set/structure.
- Inductive step: We give rules for forming new elements in the set/structure from those already known to be in the set/structure.
- Exclusion rule: We specific that all the elements in the set/structure are those elements specified in the basis step or generated by applications of the inductive step.

Remark

The exclusion rule is tacitly assumed in the slides.

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Recursively Defined Functions from Inductively Defined Sets and Structures

Example (concatenation of strings)

See whiteboard.

Recursively Defined Functions from Inductively Defined Sets and Structures

Example (concatenation of strings)

See whiteboard.

Example (length of strings)

See whiteboard.

Recursively Defined Functions from Inductively Defined Sets and Structures

Recursively defined functions from inductively defined sets and structures

A function from an inductively defined set/structure is recursively defined if:

- Basis step: We define the function on the initial collection of elements in the set/structure.

Recursively Defined Functions from Inductively Defined Sets and Structures

Recursively defined functions from inductively defined sets and structures

A function from an inductively defined set/structure is recursively defined if:

- Basis step: We define the function on the initial collection of elements in the set/structure.
- Recursive step: We give rules for defining the value of the function on a new element from those values of the function on the elements of the new element.

Recursively Defined Functions from Inductively Defined Sets and Structures

Recursively defined functions from inductively defined sets and structures

A function from an inductively defined set/structure is recursively defined if:

- Basis step: We define the function on the initial collection of elements in the set/structure.
- Recursive step: We give rules for defining the value of the function on a new element from those values of the function on the elements of the new element.

Remark

Note that the exclusion rule is not used in the definition of recursive functions.

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Structural Induction

Structural induction is used for proving properties on inductively defined sets/structures.

Structural Induction

Example

Let Σ be an alphabet and $w \in \Sigma^*$ be a string. Prove that $\lambda w = w$.

Structural Induction

Example

Let Σ be an alphabet and $w \in \Sigma^*$ be a string. Prove that $\lambda w = w$.

Proof by structural induction on w

- Basis step: Let $w = \lambda$. Then $\lambda\lambda = \lambda$ by the basis step of $(\cdot)$.

Structural Induction

Example

Let Σ be an alphabet and $w \in \Sigma^*$ be a string. Prove that $\lambda w = w$.

Proof by structural induction on w

- Basis step: Let $w = \lambda$. Then $\lambda\lambda = \lambda$ by the basis step of $(\cdot)$.
- Inductive step: Let $w = w'x$ where $w' \in \Sigma^*$ and $x \in \Sigma$.

  Inductive hypothesis (IH): $\lambda w' = w'$.

$$\begin{aligned}
\lambda w &= \lambda \cdot w'x && \text{(by def. of } w\text{)} \\
&= (\lambda \cdot w')x && \text{(by the inductive step of } (\cdot)\text{)} \\
&= w'x && \text{(by IH)} \\
&= w && \text{(by def. of } w\text{)}
\end{aligned}$$

∎

Structural Induction

Exercise

Let Σ be an alphabet and $w, w' \in \Sigma^*$ be two strings. Prove that $l(ww') = l(w) + l(w')$.

Structural Induction

Example (left and right parentheses)

Prove that every well-formed propositional formula contains an equal number of left and right parentheses.

Structural Induction

Example (left and right parentheses)

Prove that every well-formed propositional formula contains an equal number of left and right parentheses.

Proof by structural induction on the set of well-formed propositional formulae

- $P(\alpha)$: $l_\alpha = r_\alpha$, where l_α/r_α is the number of left/right parentheses in α.
- Basis step: T, F and p (propositional variable) have no parentheses, so $l_\alpha = r_\alpha = 0$, for $\alpha \in \{\mathrm{T}, \mathrm{F}, p\}$.

Continued on next slide

Structural Induction

Proof (continuation)

- Inductive step
  a) Case $\alpha = (\neg E)$:

$$\begin{aligned} l_\alpha &= 1 + l_E && \text{(def. of } \alpha\text{)} \\ &= 1 + r_E && \text{(IH in } E\text{)} \\ &= r_\alpha && \text{(def. of } \alpha\text{)} \end{aligned}$$

  b) Case $\alpha = (E * F)$:

$$\begin{aligned} l_\alpha &= 1 + l_E + l_F && \text{(def. of } \alpha\text{)} \\ &= 1 + r_E + r_F && \text{(IH in } E \text{ and } F\text{)} \\ &= r_\alpha && \text{(def. of } \alpha\text{)} \end{aligned}$$

■

Structural Induction

Structural induction

Let P be a propositional function on an inductively defined set/structure.

To prove that P is true for all the elements on the set/structure, we must make two proofs:

- Basis step: Prove P for all elements specified in the basis step of the inductive definition of the set/structure.

Structural Induction

Structural induction

Let P be a propositional function on an inductively defined set/structure.

To prove that P is true for all the elements on the set/structure, we must make two proofs:

- Basis step: Prove P for all elements specified in the basis step of the inductive definition of the set/structure.
- Inductive step: Prove that if P is true for each of the elements used to construct new elements in the inductive step of the definition, the result holds for these new elements.

Structural Induction

Problem 27(c) (p. 252)

Let S be the subset of the set of ordered pairs of integers inductively defined by

- Basis step: $(0, 0) \in S$.
- Inductive step: If $(a, b) \in S$, then
  a) $(a, b + 1) \in S$,
  b) $(a + 1, b + 1) \in S$ and
  c) $(a + 2, b + 1) \in S$.

Use structural induction to show that $a \leq 2b$ whenever $(a, b) \in S$.

Proved on next slide

Structural Induction

Proof by structural induction on S

- $P((a, b))$: If $(a, b) \in S$ then $a \leq 2b$.

Structural Induction

Proof by structural induction on S

- $P((a, b))$: If $(a, b) \in S$ then $a \leq 2b$.
- Basis step $P((0, 0))$: $0 \leq 2 \cdot 0$

Structural Induction

Proof by structural induction on S

- $P((a,b))$: If $(a,b) \in S$ then $a \leq 2b$.
- Basis step $P((0,0))$: $0 \leq 2 \cdot 0$
- Inductive step:
  a) $P((a,b+1))$:

$$\begin{aligned} a &\leq 2b && \text{(IH } P((a,b))\text{)} \\ &\leq 2b+2 && \text{(arithmetic)} \\ &\leq 2(b+1) && \text{(arithmetic)} \end{aligned}$$

Continued on next slide

Structural Induction

- Inductive step (continuation):
  b) $P((a+1, b+1))$:

$$
\begin{aligned}
a &\leq 2b && \text{(IH } P((a,b))\text{)} \\
a+1 &\leq 2b+2 && \text{(arithmetic)} \\
&\leq 2(b+1) && \text{(arithmetic)}
\end{aligned}
$$

Structural Induction

- Inductive step (continuation):

  b) $P((a+1, b+1))$:

$$\begin{aligned} a &\leq 2b && \text{(IH } P((a,b))\text{)} \\ a+1 &\leq 2b+2 && \text{(arithmetic)} \\ &\leq 2(b+1) && \text{(arithmetic)} \end{aligned}$$

  c) $P((a+2, b+1))$:

$$\begin{aligned} a &\leq 2b && \text{(IH } P((a,b))\text{)} \\ a+2 &\leq 2b+2 && \text{(arithmetic)} \\ &\leq 2(b+1) && \text{(arithmetic)} \end{aligned}$$

■

References

- Rosen, K. H. (2004). *Matemática Discreta y sus Aplicaciones*. 5th ed. Translated by José Manuel Pérez Morales and others. McGraw-Hill (cit. on p. 2).
- — (2012). Discrete Mathematics and Its Applications. 7th ed. McGraw-Hill (cit. on p. 8).