CM0246 Discrete Structures
Partial Orders

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook (Rosen 2004).

Introduction

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Some order relations.

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- Schedule projects

  $(a, b) \in R$ if a is a task that must be completed before the task b begins.

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Example

- $(\mathbb{Z}, \leq)$ is a poset.
- $(P(A), \subseteq)$ is a poset.

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Let $a, b \in \mathbb{Z}$ with $a \neq 0$. The **divisibility relation**, denoted by $|$, is defined by

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Whiteboard.

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Example

- $(\mathbb{Z}^+, |)$ is a poset.
- Is $(\mathbb{N}, |)$ a poset?

Partial Orders

Problem 6 (p. 492)

Let (A, R) be a poset. Prove that (S, R^{-1}) is also a poset, where R^{-1} is the inverse of R. The poset (S, R^{-1}) is called the dual of (S, R).

Partial Orders

Notation

$\preceq$: Denotes an arbitrary partial order

$a \prec b \stackrel{\text{def}}{=} a \preceq b \wedge a \neq b$

$(A, \preceq)$: Denotes an arbitrary poset

Comparable Elements

Definition

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Definition

If $(A, \preceq)$ is a poset and every two elements of A are comparable, A is called a **totally ordered set** (or **linearly ordered set**). The relation $\preceq$ is called a **total order** (or a **linear order**).

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- $(\mathbb{Z}, \leq)$ is a totally ordered set.
- $(\mathbb{Z}^+, |)$ is a not totally order set.
- Is $(P(A), \subseteq)$ a totally ordered set?

Well-Ordered Sets

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- $(\mathbb{N}, \leq)$ is a well-ordered set.

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- $(\mathbb{N}, \geq)$ is not a well-ordered set.

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- $(\mathbb{N}, \geq)$ is not a well-ordered set.
- Is $(\mathbb{Z}, \leq)$ a well-ordered set?

Hasse Diagrams

Example

Digraph for the relation $\{ (a, b) \mid a \leq b \}$ on $\{1, 2, 3, 4\}$.

See whiteboard.

Hasse Diagrams

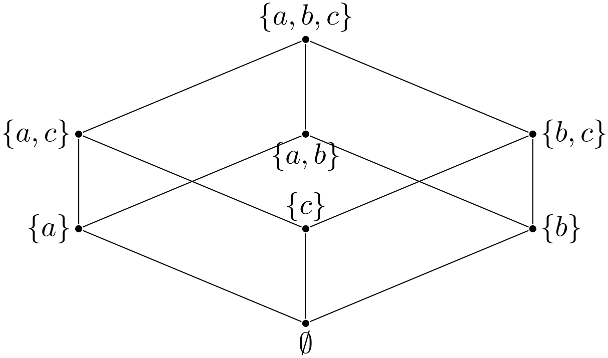
Constructing a Hasse diagram

1. Construct a digraph representation for the poset $(A, \preceq)$.
2. Remove these loops.
3. Remove all edges that must be in the partial ordering because of the presence of other edges and transitivity.
4. Arrange each edge so that its initial vertex is below its terminal vertex.
5. Remove all the arrows on the directed edges.

Hasse Diagrams

Example

Hasse diagram for the poset $(\{a, b, c\}, \subseteq)$.



Hasse Diagrams

Exercise

Draw the Hasse diagram for the poset $(\{1, 2, 3, 4, 6, 8, 12\}, |)$.

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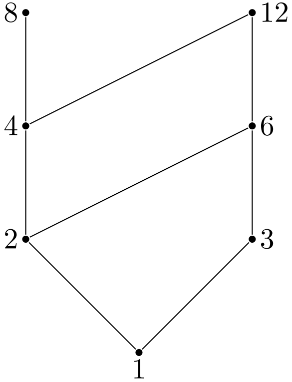

Lexicographic Ordering

Example

Let $\preceq$ be relation on $\mathbb{Z} \times \mathbb{Z}$ defined by

$$(a_1, b_1) \preceq (a_2, b_2) \overset{\text{def}}{=} a_1 < a_2 \text{ or } (a_1 = a_2 \text{ and } b_1 \leq b_2).$$

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- Is $(3, 100) \preceq (4, 4)$?

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- Is $(3, 5) \preceq (3, 4)$?
- Is $(\mathbb{Z} \times \mathbb{Z}, \preceq)$ a poset?
- Is $(\mathbb{Z} \times \mathbb{Z}, \preceq)$ a totally ordered set?

Lexicographic Ordering

Definition

Let $(A, \preceq_A)$ and $(B, \preceq_B)$ be two posets. The **lexicographic ordering** $\preceq$ on $A \times B$ is defined by:

$$(a_1, b_1) \preceq (a_2, b_2) \stackrel{\text{def}}{=} a_1 \prec_A a_2 \text{ or } (a_1 = a_2 \text{ and } b_1 \preceq_B b_2).$$

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Example

- Previous example
- Whiteboard

Lexicographic Ordering

Definition

Let $(A_1, \preceq_1), \ldots, (A_n, \preceq_n)$ be n posets. The **lexicographic ordering** $\preceq$ on $A_1 \times \cdots \times A_n$ is defined by:

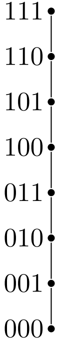
$$(a_1, \ldots, a_n) \preceq (b_1, \ldots, b_n) \stackrel{\text{def}}{=} (\exists m > 0)(\forall i < m)(a_i = b_i \land a_m \preceq_m b_m),$$

that is, if one of the terms $a_m \preceq_m b_m$ and all the preceding terms are equal.

Lexicographic Ordering

Example

Let Σ be an alphabet defined by $\Sigma = \{0, 1\}$. The lexicographical ordering on $(\Sigma, \leq) \times (\Sigma, \leq) \times (\Sigma, \leq)$ is given by



Lexicographic Ordering

Definition

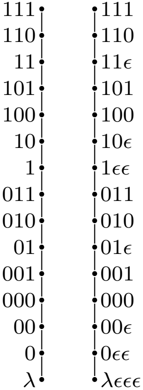
Let Σ^* be the set of all words (finite sequence of symbols) on an alphabet Σ, including the empty word denoted by λ.

A **lexicographic ordering** on Σ^* can be defined by: if the words are the same length, use the lexicographic ordering of n posets, else the shorter sequence should be padded at the end with enough "blanks" (a special symbol that is treated as smaller than every element of Σ.

Lexicographic Ordering

Example

Let Σ be an alphabet defined by $\Sigma = \{0, 1\}$. The lexicographical ordering on $\{\, w \in \Sigma^* \mid l(w) \leq 3 \,\}$ is given by



Product Order

Definition

Let $(A, \preceq_A)$ and $(B, \preceq_B)$ be two posets. The **product order** $\preceq$ on $A \times B$ is defined by:

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Example

Whiteboard.

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Problem 33 (p. 494)

Prove that the product order of two posets is a poset.

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Let $(A, \preceq_A)$ and $(B, \preceq_B)$ be two posets. We need to prove that $(A \times B, \preceq)$ is a poset, where $\preceq$ is the product order on $A \times B$.

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- Reflexivity: $(a, b) \preceq (a, b)$, for all $a \in A$ and $b \in B$. Whiteboard.

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- Antisymmetry: If $(a_1, b_1) \preceq (a_2, b_2)$ and $(a_2, b_2) \preceq (a_1, b_1)$ then $(a_1, b_1) = (a_2, b_2)$, for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Whiteboard.

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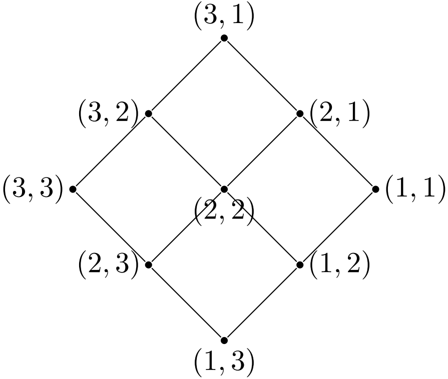
- Transitivity: If $(a_1, b_1) \preceq (a_2, b_2)$ and $(a_2, b_2) \preceq (a_3, b_3)$ then $(a_1, b_1) \preceq (a_3, b_3)$, for all $a_1, a_2, a_3 \in A$ and $b_1, b_2, b_3 \in B$. Whiteboard. ∎

Product Order

Example

Hasse diagram for the product order of the posets $(\{1,2,3\},\leq)$ and $(\{1,2,3\},\geq)$.



Notable Elements

Let $(A, \preceq)$ be a poset.

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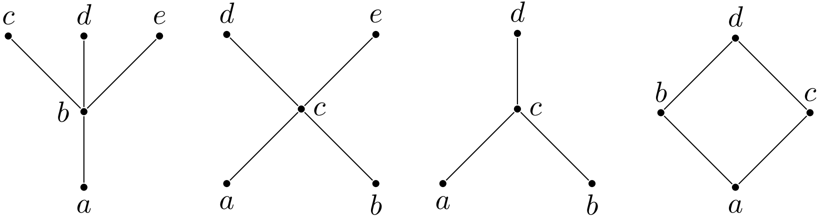
Notable Elements

Example



(a) (b) (c) (d)

Fig.	Least element	Greatest element	Maximals	Minimals
(a)	a		c, d, e	a
(b)			d, e	a, b
(c)		d	d	a, b
(d)	a	d	d	a

Notable Elements

Let $(S, \preceq)$ be a poset and let $A \subseteq S$.

Definition

Let $u \in S$ be an element such that $a \preceq u$ for all elements $a \in A$, then u is an **upper bound** of A.

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Example (using intervals of real numbers)

Whiteboard.

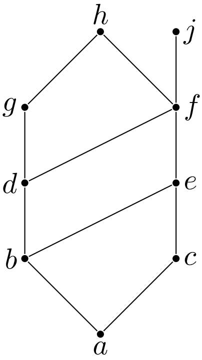
Notable Elements

Example

- $A = \{a, b, c\}$
  Upper bounds: $\{e, f, j, h\}$
  Lower bounds: $\{a\}$

- $A = \{j, h\}$
  No upper bounds.
  Lower bounds: $\{a, b, c, d, e, f\}$

- $A = \{a, c, d, f\}$
  Upper bounds: $\{f, h, j\}$
  Lower bounds: $\{a\}$



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An element x is the **supremum** (or the **least upper bound**) of the subset A, denoted by $\sup(A)$, iff x is an upper bound that is less than every other upper bound of A.

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Example (Using intervals of real numbers)

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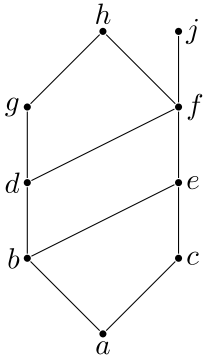
Notable Elements

Example

$A = \{b, d, g\}$

Upper bounds: $\{g, h\}$
$\sup(A) = g$
Lower bounds: $\{a, b\}$
$\inf(A) = b$

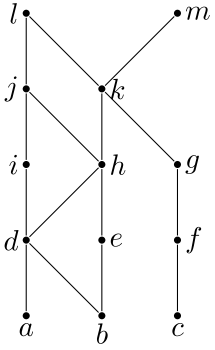


Notable Elements

Problem 26 (p. 493)

Answer these questions for the partial order represented by this Hasse diagram.

- Maximals? $\{l, m\}$
- Minimals? $\{a, b, c\}$
- Greatest element? Doesn't exist
- Least element? Doesn't exist
- Upper bounds of $\{a, b, c\}$? $\{k, l, m\}$
- $\sup(\{a, b, c\})$? k
- Lower bounds of $\{f, g, h\}$? Don't exist
- $\inf(\{f, g, h\})$? Doesn't exist

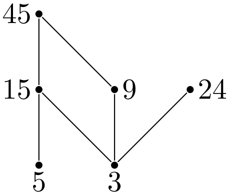


Notable Elements

Problem 27 (p. 492)

Answer these questions for the poset $(\{3, 5, 9, 15, 24, 45\}, |)$.

- Maximals? $\{24, 45\}$
- Minimals? $\{3, 5\}$
- Greatest element? Doesn't exist
- Least element? Doesn't exist
- Upper bounds of $\{3, 5\}$? $\{15, 45\}$
- $\sup(\{3, 5\})$? 15
- Lower bounds of $\{15, 45\}$? $\{3, 5, 15\}$
- $\inf(\{15, 45\})$? 15



References

Rosen, K. H. (2004). *Matemática Discreta y sus Aplicaciones*. 5th ed. Translated by José Manuel Pérez Morales and others. McGraw-Hill (cit. on p. 2).