CM0246 Discrete Structures
Mathematical Induction

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook (Rosen 2004).

Motivation

Exercise

Conjecture a formula for the sum of the first n positive odd integers.

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Question

Let $P(n)$ be a propositional function. How can we proof that $P(n)$ is true for all $n \in \mathbb{Z}^+$?

Principle of Mathematical Induction

Proof by mathematical induction

Let $P(n)$ be a propositional function.

To prove that $P(n)$ is true for all $n \in \mathbb{Z}^+$, we must make two proofs:

- Basis step: Prove $P(1)$

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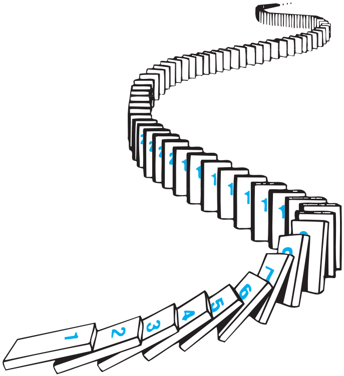
- Basis step: Prove $P(1)$
- Inductive step: Prove $P(k) \to P(k+1)$ for all $k \in \mathbb{Z}^+$

  $P(k)$ is called the inductive hypothesis.

Principle of Mathematical Induction

How mathematical induction works†



†Figure source: (Rosen 2012, § 5.1, Fig. 2).

Principle of Mathematical Induction

Definition (principle of mathematical induction)

Inference rule version:

$$\dfrac{P(1) \qquad \begin{array}{c}[P(k)] \\ \vdots \\ P(k+1)\end{array}}{P(n)} \text{ (PMI)}$$

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Axiom (or theorem) version: Let P be a propositional function (predicate). Then

$$[P(1) \wedge \forall k(P(k) \rightarrow P(k+1)] \rightarrow \forall n P(n) \quad \text{(PMI)}$$

Principle of Mathematical Induction

Methodology for proving by mathematical induction

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3. Prove the induction step, i.e. $\forall k(P(k) \to P(k+1))$.

   Remark: In this proof you need to use the inductive hypothesis $P(n)$.

Principle of Mathematical Induction

Methodology for proving by mathematical induction

1. State the propositional function $P(n)$.
2. Prove the basis step, i.e. $P(1)$.
3. Prove the induction step, i.e. $\forall k(P(k) \rightarrow P(k+1))$.

   Remark: In this proof you need to use the inductive hypothesis $P(n)$.
4. Conclude $\forall n P(n)$ by the principle of mathematical induction.

Principle of Mathematical Induction

Example

Prove that the sum of the first n odd positive integers is n^2.[†]

Whiteboard.

[†] Historical remark. From 1575, it could be the first property proved using the PMI (Gunderson 2011, § 1.8).

Principle of Mathematical Induction

Example

Prove that if $n \in \mathbb{Z}^+$, then

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Principle of Mathematical Induction

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Proof

1. $P(n)$: $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Principle of Mathematical Induction

Example

Prove that if $n \in \mathbb{Z}^+$, then

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof

1. $P(n)$: $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.
2. Basis step $P(1)$: $1 = \frac{1(1+1)}{2}$.

Continued on next slide

Principle of Mathematical Induction

Proof (continuation)

3. Inductive step:

   Inductive hypothesis $P(k)$: $1 + 2 + 3 + \cdots + k = \dfrac{k(k+1)}{2}$.

   Let's prove $P(k+1)$:

$$
\begin{aligned}
1 + 2 + 3 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) && \text{(by IH)} \\
&= (k+1)\left(\frac{k}{2} + 1\right) && \text{(by arithmetic)} \\
&= \frac{(k+1)(k+2)}{2} && \text{(by arithmetic)}
\end{aligned}
$$

Principle of Mathematical Induction

Proof (continuation)

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\end{aligned}
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4. $\forall n P(n)$ by the principle of induction mathematical. ■

Principle of Mathematical Induction

Example

Prove that if $n \in \mathbb{N}$, then

$$2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

Proved on next slide

Principle of Mathematical Induction

Proof

1. $P(n)$: $2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$

Principle of Mathematical Induction

Proof

1. $P(n)$: $2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$
2. Basis step $P(0)$: $2^0 = 1 = 2^{0+1} - 1$.

Principle of Mathematical Induction

Proof

1. $P(n)$: $2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$
2. Basis step $P(0)$: $2^0 = 1 = 2^{0+1} - 1$.
3. Inductive step:
   Inductive hypothesis $P(k)$: $2^0 + 2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$
   Let's prove $P(k+1)$:

$$
\begin{aligned}
2^0 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} &= 2^{k+1} - 1 + 2^{k+1} && \text{(by IH)} \\
&= 2(2^{k+1}) - 1 && \text{(by arithmetic)} \\
&= 2^{k+2} - 1 && \text{(by arithmetic)}
\end{aligned}
$$

Principle of Mathematical Induction

Proof

1. $P(n)$: $2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$
2. Basis step $P(0)$: $2^0 = 1 = 2^{0+1} - 1$.
3. Inductive step:
   Inductive hypothesis $P(k)$: $2^0 + 2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$
   Let's prove $P(k+1)$:

$$\begin{aligned} 2^0 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} &= 2^{k+1} - 1 + 2^{k+1} && \text{(by IH)} \\ &= 2(2^{k+1}) - 1 && \text{(by arithmetic)} \\ &= 2^{k+2} - 1 && \text{(by arithmetic)} \end{aligned}$$

4. $\forall n P(n)$ by the principle of induction mathematical. ■

Strong Induction

Proof by strong (or course-of-values) induction

Let $P(n)$ be a propositional function.

To prove that $P(n)$ is true for all $n \in \mathbb{Z}^+$, we must make two proofs:

- Basis step: Prove $P(1)$

Strong Induction

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To prove that $P(n)$ is true for all $n \in \mathbb{Z}^+$, we must make two proofs:

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Let $P(n)$ be a propositional function.

To prove that $P(n)$ is true for all $n \in \mathbb{Z}^+$, we must make two proofs:

- Basis step: Prove $P(1)$
- Inductive step: Prove $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ for all $k \in \mathbb{Z}^+$

  The (strong) inductive hypothesis is given by

  $$P(j) \text{ is true for } j = 1, 2, \ldots, k.$$

Strong Induction

Definition ([strong induction)

Inference rule version:

$$\frac{P(1) \qquad \forall k[(P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \rightarrow P(k+1)]}{\forall n P(n)} \text{ (strong induction)}$$

Strong Induction

Example (a part of the fundamental theorem of arithmetic)

Prove that if n is an integer greater than 1, either is prime itself or is the product of prime numbers.

Proved on next slide

Strong Induction

Proof

1. $P(n)$: n is prime itself or it is the product of prime numbers.

Strong Induction

Proof

1. $P(n)$: n is prime itself or it is the product of prime numbers.
2. Basis step $P(2)$: 2 is a prime number.

Strong Induction

Proof

1. $P(n)$: n is prime itself or it is the product of prime numbers.
2. Basis step $P(2)$: 2 is a prime number.
3. Inductive step:
   Inductive hypothesis: $P(j)$ is true for $j = 1, 2, \ldots, k$.

Strong Induction

Proof

1. $P(n)$: n is prime itself or it is the product of prime numbers.
2. Basis step $P(2)$: 2 is a prime number.
3. Inductive step:
   Inductive hypothesis: $P(j)$ is true for $j = 1, 2, \ldots, k$.
   Let's prove that $k + 1$ satisfies the property:

Strong Induction

Proof

1. $P(n)$: n is prime itself or it is the product of prime numbers.
2. Basis step $P(2)$: 2 is a prime number.
3. Inductive step:
   Inductive hypothesis: $P(j)$ is true for $j = 1, 2, \ldots, k$.
   Let's prove that $k + 1$ satisfies the property:
   3.1 If $k + 1$ is a prime number then it satisfies the property.

Strong Induction

Proof

1. $P(n)$: n is prime itself or it is the product of prime numbers.
2. Basis step $P(2)$: 2 is a prime number.
3. Inductive step:
   Inductive hypothesis: $P(j)$ is true for $j = 1, 2, \dots, k$.
   Let's prove that $k + 1$ satisfies the property:
   3.1 If $k + 1$ is a prime number then it satisfies the property.
   3.2 If $k + 1$ is a composite number:
   $k + 1 = ab$ where $2 \leq a \leq b < k + 1$. Since $P(a)$ and $P(b)$ by the inductive hypothesis, then $P(k + 1)$.

Strong Induction

Proof

1. $P(n)$: n is prime itself or it is the product of prime numbers.
2. Basis step $P(2)$: 2 is a prime number.
3. Inductive step:
   Inductive hypothesis: $P(j)$ is true for $j = 1, 2, \ldots, k$.
   Let's prove that $k + 1$ satisfies the property:
   3.1 If $k + 1$ is a prime number then it satisfies the property.
   3.2 If $k + 1$ is a composite number:
   $k + 1 = ab$ where $2 \leq a \leq b < k + 1$. Since $P(a)$ and $P(b)$ by the inductive hypothesis, then $P(k + 1)$.
4. $P(n)$ is true for all integer n greater than 1 by strong induction. ■

First-Order Peano Arithmetic

Giuseppe Peano
(1858 – 1932)

Axioms of first-order Peano arithmetic[†]

$$\forall n.\ 0 \neq n'$$
$$\forall m \forall n.\ m' = n' \to m = n$$
$$\forall n.\ 0 + n = n$$
$$\forall m \forall n.\ m' + n = (m + n)'$$
$$\forall n.\ 0 * n = 0$$
$$\forall m \forall n.\ m' * n = n + (m * n)$$

[†]See, for example, (Hájek and Pudlák 1998).

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For all formulae A,

$$[A(0) \wedge (\forall n.\ A(n) \rightarrow A(n'))] \rightarrow \forall n A(n)$$

[†]See, for example, (Hájek and Pudlák 1998).

First-Order Peano Arithmetic

Theorem

The principle of mathematical induction and strong induction are equivalent.[†]

[†] See, for example, (Gunderson 2011).

References

- Gunderson, D. S. (2011). Handbook of Mathematical Induction. Chapman & Hall (cit. on pp. 15, 40).
- Hájek, P. and Pudlák, P. (1998). Metamathematics of First-Order Arithmetic. Second printing. Springer (cit. on pp. 38, 39).
- Rosen, K. H. (2004). *Matemática Discreta y sus Aplicaciones*. 5th ed. Translated by José Manuel Pérez Morales and others. McGraw-Hill (cit. on p. 2).
- — (2012). Discrete Mathematics and Its Applications. 7th ed. McGraw-Hill (cit. on p. 7).