CM0246 Discrete Structures
Lattices

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Semester 2014-2

Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook (Rosen 2004).

Lattices from the Partial Orders Theory

Definition

A **lattice** (*retículo*) is a poset where every pair of elements has both a supremum and an infimum.

Lattices from the Partial Orders Theory

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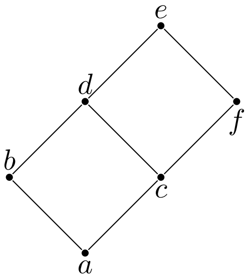
Example

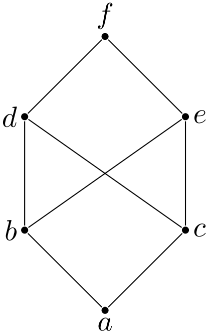
The following poset is a lattice.



Lattices from the Partial Orders Theory

Example (counter-example)

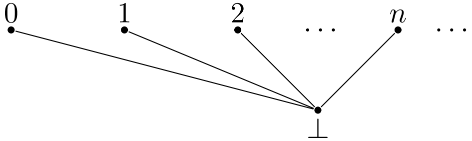
The following poset is not a lattice because the upper bounds of the pair $\{b, c\}$ are d, e and f, but this set has not a least upper bound.



Lattices from the Partial Orders Theory

Example (counter-example)

The following poset is not a lattice because for example, the pair $\{1, 2\}$ has not supremum.



Lattices from the Partial Orders Theory

Example

- $(\mathbb{Z}^+, |)$ is a lattice where the supremum is the least common multiple and the infimum is the greatest common divisor.

Lattices from the Partial Orders Theory

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- $(\mathbb{Z}^+, |)$ is a lattice where the supremum is the least common multiple and the infimum is the greatest common divisor.
- Let A be a set. Is $(P(A), \subseteq)$ a lattice?

Algebraic Structures

Definition

An **algebraic structure** on a set $A \neq 0$ is essentially a collection of n-ary operations on A (Cohn 1981, p. 41).

Example

A **semigroup** $(S, *)$ is a set S with an associative binary operation $* : S \times S \to S$.

Example

A **monoid** $(M, *, \epsilon)$ is a semigroup $(M, *)$ with an element $\epsilon \in M$ which is an unit for $*$, i.e. $\forall x(x * \epsilon = \epsilon * x = x)$.

Lattices from the Algebraic Structures Theory

Definition

Let $\wedge$ and $\vee$ be two binaries operations, called meet and join, respectively. A **lattice** retículo is an algebraic structure $(L, \wedge, \vee)$, which satisfy the following axioms for all x, y and z in L (Lipschutz and Lipson 2007):

$$x \wedge y = y \wedge x \qquad \text{(Commutative laws)}$$
$$x \vee y = y \vee x$$

$$(x \wedge y) \wedge z = x \wedge (y \wedge z) \qquad \text{(Associative laws)}$$
$$(x \vee y) \vee z = x \vee (y \vee z)$$

$$x \wedge (x \vee y) = x \qquad \text{(Absortion laws)}$$
$$x \vee (x \wedge y) = x$$

Lattices from the Algebraic Structures Theory

Example

Let A be a set. $(P(A), \cap, \cup)$ is a lattice.

Lattices from the Algebraic Structures Theory

Definition

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Theorem (principle of duality)

The dual of any theorem in a lattice is also an theorem (Lipschutz and Lipson 2007).

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Theorem (principle of duality)

The dual of any theorem in a lattice is also an theorem (Lipschutz and Lipson 2007).

Proof.

The dual of every axiom in a lattice is also an axiom. Hence, the dual theorem can be proved by using the dual of each step of the proof of the original theorem. ■

Lattices from the Algebraic Structures Theory

Example

Let $(L, \wedge, \vee)$ be a lattice. Prove the idempotent laws

$$x \wedge x = x, \tag{1}$$

$$x \vee x = x. \tag{2}$$

Lattices from the Algebraic Structures Theory

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Let $(L, \wedge, \vee)$ be a lattice. Prove the idempotent laws

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Proof of (1).

$$\begin{aligned} x \wedge x &= x \wedge (x \vee (x \wedge y)) && \text{(second absorption law)} \\ &= x && \text{(first absorption law)} \end{aligned}$$

■

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Proof of (2).

By principle of duality on (1). ■

Lattices from the Algebraic Structures Theory

Problem 40 (p. 500)

Prove that if x and y are elements of a lattice $(L, \wedge, \vee)$ then $x \vee y = y$, if and only if, $x \wedge y = x$.

Lattices from the Algebraic Structures Theory

Problem 40 (p. 500)

Prove that if x and y are elements of a lattice $(L, \wedge, \vee)$ then $x \vee y = y$, if and only if, $x \wedge y = x$.

Proof $\rightarrow$.

Let's suppose $x \vee y = y$. Then

$$\begin{aligned} x &= x \wedge (x \vee y) && \text{(first absorption law)} \\ &= x \wedge y && \text{(hypothesis)} \end{aligned}$$

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Lattices from the Algebraic Structures Theory

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■

Continued on next slide

Lattices from the Algebraic Structures Theory

Proof $\leftarrow$.

Let's suppose $x \wedge y = x$. Then

$$
\begin{aligned}
y &= y \vee (y \wedge x) && \text{(second absorption law)} \\
&= y \vee (x \wedge y) && \text{(commutative law)} \\
&= y \vee x && \text{(hypothesis)} \\
&= x \vee y && \text{(commutative law)}
\end{aligned}
$$

■

Equivalence of the Definitions

Theorem

Let $(L, \wedge, \vee)$ be a lattice. Then $(L, \preceq)$ is a partial order, where the relation $\preceq$ is defined by (Lipschutz and Lipson 2007):

$$x \preceq y \stackrel{\text{def}}{=} x \wedge y = x.$$

Equivalence of the Definitions

Theorem

Let $(L, \wedge, \vee)$ be a lattice. Then $(L, \preceq)$ is a partial order, where the relation $\preceq$ is defined by (Lipschutz and Lipson 2007):

$$x \preceq y \stackrel{\text{def}}{=} x \wedge y = x.$$

Proof.

1. The relation $\preceq$ is reflexive

   $x \wedge x = x$ (idempotency), for all $x \in L$. Therefore $x \preceq x$, for all $x \in L$.

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Equivalence of the Definitions

Proof (continuation)

2. The relation $\preceq$ is antisymmetric

   Suppose $x \preceq y$ and $y \preceq x$, then $x \wedge y = x$ and $y \wedge x = y$. Therefore

$$
\begin{aligned}
x &= x \wedge y && \text{(hypothesis)} \\
&= y \wedge x && \text{(commutative law)} \\
&= y && \text{(hypothesis)}
\end{aligned}
$$

   That is, $\preceq$ is antisymmetric.

Continued on next slide

Equivalence of the Definitions

Proof (continuation).

3. The relation $\preceq$ is transitive

   Suppose $x \preceq y$ and $y \preceq z$, then $x \wedge y = x$ and $y \wedge z = y$. Therefore

$$
\begin{aligned}
x \wedge z &= (x \wedge y) \wedge z && \text{(hypothesis)} \\
&= x \wedge (y \wedge z) && \text{(associativity law)} \\
&= x \wedge y && \text{(hypothesis)} \\
&= x && \text{(hypothesis)}
\end{aligned}
$$

   That is, $x \preceq z$. ■

Equivalence of the Definitions

Remark

Let $(L, \wedge, \vee)$ be a lattice and let be $(L, \preceq)$ the order partial induced by $(L, \wedge, \vee)$. It is possible prove that $(L, \preceq)$ is a lattice.

Equivalence of the Definitions

Theorem (Problem 39, p. 500)

Let $(L, \preceq)$ be a lattice. Then $(L, \wedge, \vee)$ is a lattice, where

$$x \wedge y \stackrel{\text{def}}{=} \inf(x, y),$$
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Theorem (Problem 39, p. 500)

Let $(L, \preceq)$ be a lattice. Then $(L, \wedge, \vee)$ is a lattice, where

$$x \wedge y \overset{\text{def}}{=} \inf(x, y),$$
$$x \vee y \overset{\text{def}}{=} \sup(x, y),$$

Proof.

1. Commutative laws for $\wedge$ and $\vee$ (Rosen's solution).

   Because $\inf(x, y) = \inf(y, x)$ and $\sup(x, y) = \sup(y, x)$, it follows that $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.

Continued on next slide

Equivalence of the Definitions

Proof (continuation)

2. Associative laws for $\wedge$ and $\vee$ (Rosen's solution).

   Using the definition, $(x \wedge y) \wedge z$ is a lower bound of x, y and z that is greater than every other lower bound. Because x, y and z play interchangeable roles, $x \wedge (y \wedge z)$ is the same element.

Equivalence of the Definitions

Proof (continuation)

2. Associative laws for $\wedge$ and $\vee$ (Rosen's solution).

   Using the definition, $(x \wedge y) \wedge z$ is a lower bound of x, y and z that is greater than every other lower bound. Because x, y and z play interchangeable roles, $x \wedge (y \wedge z)$ is the same element.

   Similarly, $(x \vee y) \vee z$ is an upper bound of x, y and z that is less than every other upper bound. Because x, y and z play interchangeable roles, $x \vee (y \vee z)$ is the same element.

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Equivalence of the Definitions

Proof (continuation).

3. Absorption laws for $\wedge$ and $\vee$ (Rosen's solution).

   To show that $x \wedge (x \vee y) = x$ it is sufficient to show that x is the greatest lower bound of x, and $x \vee y$. Note that x is a lower bound of x, and because $x \vee y$ is by definition greater than x, x is a lower bound for it as well. Therefore, x is a lower bound. But any lower bound of x has to be less than x, so x is the greatest lower bound.

   The second statement is the dual of the first; we omit its proof. ■

References

- Cohn, P. M. (1981). Universal Algebra. Revised edition. D. Reidel Publishing Company (cit. on p. 9).
- Lipschutz, S. and Lipson, M. L. (2007). Schaum's Outline of Discrete Mathematics. 3rd ed. McGraw-Hill (cit. on pp. 10, 12–15, 23, 24).
- Rosen, K. H. (2004). *Matemática Discreta y sus Aplicaciones*. 5th ed. Translated by José Manuel Pérez Morales and others. McGraw-Hill (cit. on p. 2).