CM0246 Discrete Structures
Functions

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook (Rosen 2004).

Preliminaries

Remark

Recall that if an element of a set is listed more than once it doesn't matter.

Example

$$\{1, 3, 3, 3, 5, 5, 5, 5\} = \{1, 3, 5\}.$$

Preliminaries

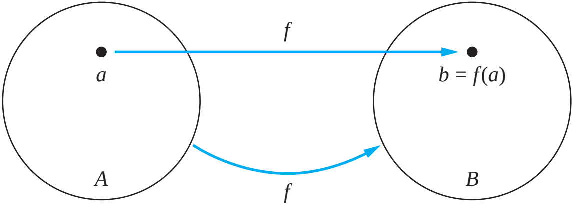
Notation

$\mathbb{N} = \{0, 1, 2, \ldots\}$	(natural numbers, non-negative integers)
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$	(integers)
$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$	(positive integers)
$\mathbb{Q} = \{\, p/q \mid p, q \in \mathbb{Z} \text{ and } q \neq 0 \,\}$	(rational numbers)
$\mathbb{R} = (-\infty, \infty)$	(real numbers)

Functions

Definition

Let A and B be sets. A **function** (**map**, **mapping** or **transformation**) f from A to B, denoted $f : A \to B$, is an assignment of exactly one element of B to each element of A.[†]



[†]Figure source: (Rosen 2012, § 2.3, Fig. 2).

Functions

Specification of functions

- Explicitly

Functions

Specification of functions

- Explicitly
- Formula

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Specification of functions

- Explicitly
- Formula
- Programming languages

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(Advance) question

Are all the functions define using a programming language really functions?

Functions

Definitions 1

Let f be a function from A to B:

Domain of f: A

Codomain of f: B

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If $f(a) = b$:

b is the **image** of a

a is a **preimage** of b

The **range** or **image** of f: Set of all images of elements of A

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If $f(a) = b$:

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The **range** or **image** of f: Set of all images of elements of A

If S is a subset of A: $f(S) = \{ f(s) \mid s \in S \}$

Functions

Example

See slides § 2.3, p. 2 for the 6th ed. of Rosen's textbook.

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Problem 3 (p. 99)

Determine whether f is a function from the set of all bit strings to the set of integers if

- $f(s)$ is the number of 1 bits in s

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Problem 3 (p. 99)

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Problem 3 (p. 99)

Determine whether f is a function from the set of all bit strings to the set of integers if

- $f(s)$ is the number of 1 bits in s
- $f(s)$ is the position of a 0 bit in s
- $f(s)$ is the smallest integer i such that the ith bit of s is 1 and $f(s) = 0$ when s is the empty string

Functions

Definition

Let A and B be sets. The **Cartesian product** of A and B is

$$A \times B = \{\, (a, b) \mid a \in A \land b \in B \,\}\,.$$

Example

Let $A = \{a, b\}$ and $B = \{1, 2\}$. Then

$$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}.$$

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Definition

A subset R of the Cartesian product $A \times B$ is called a **relation** from the set A to the set B.

Functions

Definition

Let A and B be sets. A **function** f from A to B is a relation of A to B (i.e. subset of $A \times B$) such that

$$\forall x(x \in A \rightarrow \exists y(y \in B \wedge (x, y) \in f))$$

and

$$\forall x \forall y \forall y' \{((x, y) \in f \wedge (x, y') \in f) \rightarrow y = y'\}.$$

Remark

In some theories different to set theory, the concept of function is a primitive concept.

Injective Functions

Definition

Let $f : A \to B$. The function f is an **injunction** (or **one-to-one**),

if and only if,

$f(a) = f(a')$ implies that $a = a'$ for all $a, a' \in A$.

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Example

Whiteboard.

Injective Functions

Definition

Let $f : A \to B$. The function f is an **injunction**, if and only if,

$$\forall x \forall x' (f(x) = f(x') \to x = x')$$

or equivalent

$$\forall x \forall x' (x \neq x' \to f(x) \neq f(x')) \quad \text{(contrapositive)}$$

where A is the domain of quantification.

Injective Functions

Is a function (non)-injective?

Let $f : A \to B$.

- To show that f is injective

  Show that if $f(x) = f(x')$ for arbitrary $x, x' \in A$ then $x = x'$.

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  Find particular elements $x, x' \in A$ such that $x \neq x'$ and $f(x) = f(x')$.

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Exercise

Is the function $f : \mathbb{Z} \to \mathbb{Z}$, where $f(x) = x^2$ injective?

Surjective Functions

Definition

Let $f : A \to B$. The function f is a **surjection** (or **onto**),

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for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

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Example

Whiteboard.

Surjective Functions

Definition

Let $f : A \to B$. The function f is a **surjection**, if and only if,

$$\forall y \exists x (f(x) = y),$$

where A is the domain of quantification for x and B is the domain of quantification for y.

Surjective Functions

Is a function (non)-surjective?

Let $f : A \to B$.

- To show that f is surjective

  Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

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  Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

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Exercise

Is the function $f : \mathbb{Z} \to \mathbb{Z}$, where $f(x) = x + 1$ surjective?

Bijective Functions

Definition

A function f is a **bijection** (or **one-to-one correspondence**),

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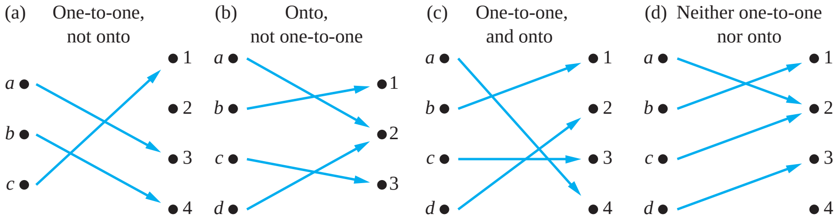
Example (the identity function)

Let A be a set. The identity function $\iota_A : A \to A$ where $\iota_A(a) = a$ is bijective.

Injective, Surjective and Bijective Functions

Example

Types of functions.[†]



[†] Figure source: (Rosen 2012, § 2.3, Fig. 5).

Injective, Surjective and Bijective Functions

Problem 16 (p. 100)

Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$:

a) injective but not surjective

$$f(x) = x + 1.$$

b) surjective but not injective

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \text{ or } x = 1; \\ x - 1, & \text{otherwise.} \end{cases}$$

Continued on next slide

Injective, Surjective and Bijective Functions

Problem (continuation)

c) bijective (but different from the identity function)

$$f(x) = \begin{cases} 0, & \text{if } x = 1; \\ 1, & \text{if } x = 0; \\ x, & \text{otherwise.} \end{cases}$$

d) neither injective nor surjective

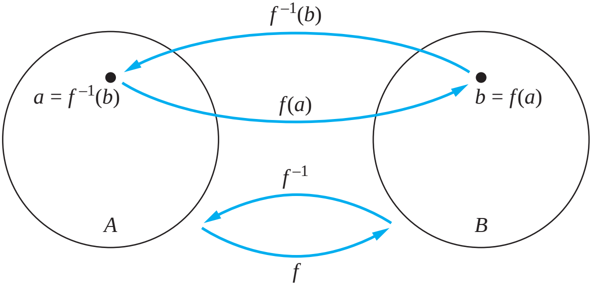
$$f(x) = 5.$$

Inverse Functions

Definition

Let $f : A \to B$ be a bijective function. The **inverse function** of f, denoted f^{-1}, is the function from B to A defined by[†]

$$f^{-1}(b) = a \text{ iff } f(a) = b.$$



[†]Figure source: (Rosen 2012, § 2.3, Fig. 6).

Inverse Functions

Exercise

Let $f : \mathbb{Z} \to \mathbb{Z}$, where $f(x) = x + 1$. Find f^{-1}.

Inverse Functions

Exercise

Let $f : \mathbb{Z} \to \mathbb{Z}$, where $f(x) = x + 1$. Find f^{-1}.

Question

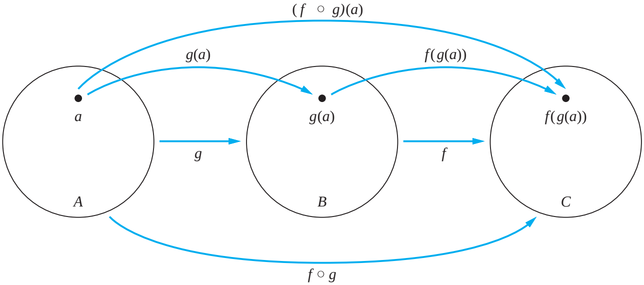
If a function f is not bijective, we cannot define f^{-1}. Why?

Composition of Functions

Definition

Let $g : A \to B$ and $f : B \to C$ be two functions. The **composition** of f with g, denoted $f \circ g$, is the function from A to C defined by†

$$(f \circ g)(a) = f(g(a)).$$



†Figure source: (Rosen 2012, § 2.3, Fig. 7).

Composition of Functions

Example

See slides § 2.3, p. 8 for the 6th ed. of Rosen's textbook.

Composition of Functions

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Remark

Let f and g be functions. The composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.

Composition of Functions

Example

Let $f : A \to B$ a bijective function such that $f(a) = b$.

The function $f \circ f^{-1} : B \to B$ is defined by

$$\begin{aligned}
(f \circ f^{-1})(b) &= f(f^{-1}(b)) && \text{(by def. of composition)} \\
&= f(a) && \text{(by def. of inverse function)} \\
&= b && \text{(by def. of inverse function)}
\end{aligned}$$

that is, $f \circ f^{-1} = \iota_B$.

Composition of Functions

Remark

In general, the composition of function is not commutative.

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Exercise

Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions, where $f(x) = x^2$ and $g(x) = 2x + 1$. Show that $f \circ g \neq g \circ f$.

Composition of Functions

Problem 25(b) (p. 100)

Let $g : A \to B$ and $f : B \to C$ be two functions. Show that if both f and g are surjective functions, then $f \circ g$ is also surjective.

Solved on next slide

Composition of Functions

Proof

1. Let $c \in C$.

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2. $f(b) = c$, for some $b \in B$ because f is surjective by hypothesis.

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Proof

1. Let $c \in C$.
2. $f(b) = c$, for some $b \in B$ because f is surjective by hypothesis.
3. $g(a) = b$, for some $a \in A$ because g is surjective by hypothesis.
4. Then

$$\begin{aligned}
(f \circ g)(a) &= f(g(a)) && \text{(by def. of composition)} \\
&= f(b) && \text{(by 3)} \\
&= c. && \text{(by 2)}
\end{aligned}$$

Composition of Functions

Proof

1. Let $c \in C$.
2. $f(b) = c$, for some $b \in B$ because f is surjective by hypothesis.
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5. That is, for all $c \in C$, exists $a \in A$ such that $(f \circ g)(a) = c$.

Composition of Functions

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4. Then

$$\begin{aligned}(f \circ g)(a) &= f(g(a)) && \text{(by def. of composition)}\\ &= f(b) && \text{(by 3)}\\ &= c. && \text{(by 2)}\end{aligned}$$

5. That is, for all $c \in C$, exists $a \in A$ such that $(f \circ g)(a) = c$.
6. Hence, $f \circ g$ is surjective. ∎

Composition of Functions

Problem 26 (p. 101)

If f and $f \circ g$ are injections, does it follow that g is injective? Justify your answer.

Composition of Functions

Proof

The function g must be injective.

1. Let $g : A \to B$ and $f : B \to C$ be two functions, and suppose g is not injective.
2. Exists distinct elements $x, x' \in A$ such that $g(x) = g(x')$ because g is not injective.
3. Then

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{(by def. de composition)} \\ &= f(g(x')) && \text{(by step 2)} \\ &= (f \circ g)(x'). && \text{(by def. de composition)} \end{aligned}$$

   Hence, $f \circ g$ is not injective (contradiction).
4. Therefore, the function g must be injective. ∎

The Graphs of Functions

Definition

Let $f : A \to B$. The **graph** of f is the set

$$\{ (a, b) \mid a \in A \text{ and } f(a) = b \}.$$

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Example

Whiteboard

The Graphs of Functions

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Example

Whiteboard

Remarks

- Graphs of functions and polymorphic functions

The Graphs of Functions

Definition

Let $f : A \to B$. The **graph** of f is the set

$$\{ (a, b) \mid a \in A \text{ and } f(a) = b \} .$$

Example

Whiteboard

Remarks

- Graphs of functions and polymorphic functions
- Graphs of functions and programs

The Graphs of Functions

Definition

Let $f : A \to B$. The **graph** of f is the set

$$\{\, (a, b) \mid a \in A \text{ and } f(a) = b \,\}\,.$$

Example

Whiteboard

Remarks

- Graphs of functions and polymorphic functions
- Graphs of functions and programs
- Partial and total functions

References

- Rosen, K. H. (2004). *Matemática Discreta y sus Aplicaciones*. 5th ed. Translated by José Manuel Pérez Morales and others. McGraw-Hill (cit. on p. 2).
- — (2012). Discrete Mathematics and Its Applications. 7th ed. McGraw-Hill (cit. on pp. 5, 34, 37, 40).