CM0246 Discrete Structures
Euler and Hamilton Paths

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Preliminaries

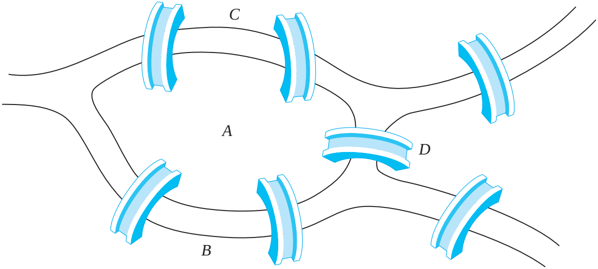
Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook (Rosen 2004).

The Problem of the Seven Bridges of Königsberg

Problem

It is possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point?[†]



[†]Figure source: (Rosen 2012, § 10.5, Fig. 1).

Paths

Definition

Let n be a non-negative integer and G an undirected graph. A **path** of length n from u to v in G is a sequence of n edges $e_1, e_2, \ldots, e_n$ of G such that $f(e_1) = \{x_0, x_1\}$, $f(e_2) = \{x_1, x_2\}$, ..., $f(e_n) = \{x_{n-1}, x_n\}$, where $x_0 = u$ and $x_n = v$.

Example

Whiteboard.

Paths

Definition

A path is a **circuit** if $u = v$ (it begins and ends at the same vertex) and has length greater than zero.

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Definition

A path/circuit is **simple** if it does not contain the same edge more than once.

Example

Whiteboard.

Connectedness

Definition

An undirected graph is called **connected** (*conexo*) if there is a path between every pair of distinct vertices of the graph.

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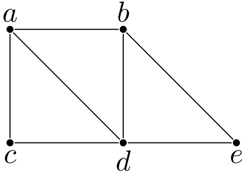
Exercise

Find an Euler path in the following graph:



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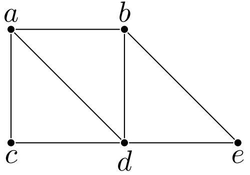

Solution

An Euler path is a, c, d, e, b, d, a, b.

Euler Circuits

Definition

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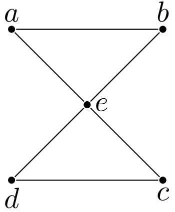

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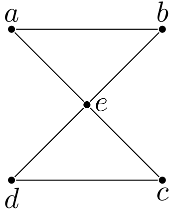


Solution

An Euler circuit is a, e, c, d, e, b, a.

Euler Paths and Circuits

Theorem 1 (p. 543)

A connected multigraph with at least two vertices has an Euler circuit, if and only if, each of its vertices has even degree.

Euler Paths and Circuits

Example (solution to the problem of the seven bridges of Königsberg)

It is possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point?

† Figure source: (Rosen 2012, § 10.5, Fig. 1).

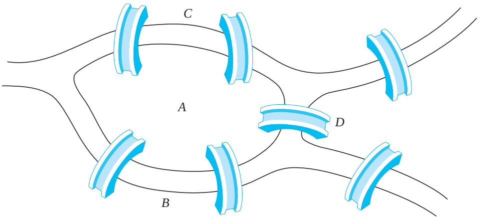
Euler Paths and Circuits

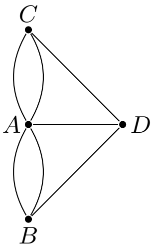
Example (solution to the problem of the seven bridges of Königsberg)

It is possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point?

Solution

No, it is not possible, because the multigraph representing the bridges has (four) vertices of odd degree.[†]





[†]Figure source: (Rosen 2012, § 10.5, Fig. 1).

Euler Paths and Circuits

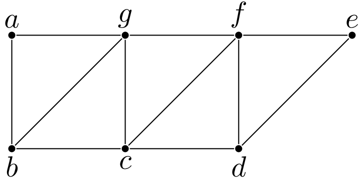
Theorem 2 (p. 544)

A connected multigraph has an Euler path but not an Euler circuit, if and only if it has exactly two vertices of odd degree.

Euler Paths and Circuits

Exercise

Has the following graph an Euler path? If so, find it.



Euler Paths and Circuits

Exercise

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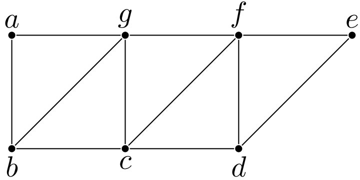


Solution

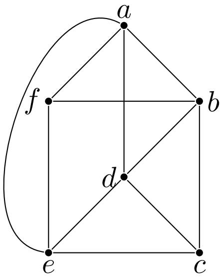
The only vertices with degree odd are b and d, so the graph has an Euler path. An Euler path is

$$b, a, g, f, e, d, c, g, b, c, f, d.$$

Euler Paths and Circuits

Problem 4 (p. 550)

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



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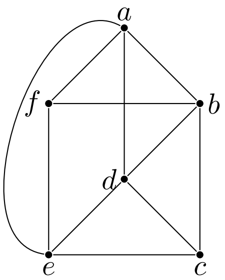
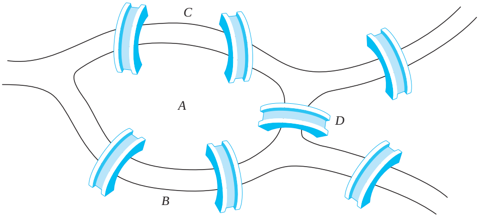


- The graph has no an Euler circuit because it has at least a vertice of odd degree ($\delta(f) = 3$).
- The graph has an Euler path because the only two vertices of odd degree are f and c.
- Euler path: $f, a, e, f, b, a, d, e, c, d, b, c$.

Euler Paths and Circuits

Problem 9 (p. 550)

Suppose that in addition to the seven bridges of Königsberg there were two additional bridges, connecting regions B and C and regions B and D, respectively. Could someone cross all nine of these bridges exactly once and return to the starting point?†



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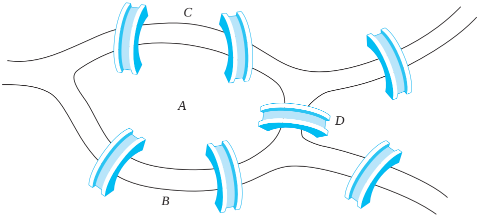
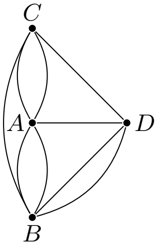




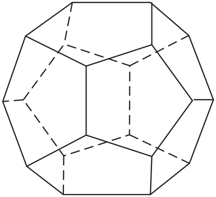
The problem asks for building an Euler circuit. It it not possible because the graph representing the problem has vertices of odd degree (e.g. $\delta(A) = 5$).

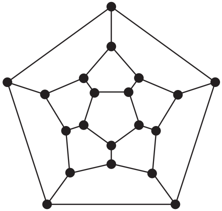
†Figure source: (Rosen 2012, § 10.5, Fig. 1).

The Icosian Puzzle

The icosian puzzle

‘The icosian puzzle (invented by William Rowan Hamilton) consisted of a wooden dodecahedron (a polyhedron with 12 regular pentagons as faces), with a peg at each vertex of the dodecahedron, and string. The 20 vertices of the dodecahedron were labeled with different cities in the world. The object of the puzzle was to start at a city and travel along the edges of the dodecahedron, visiting each of the other 19 cities exactly once, and end back at the first city.’ (Rosen 2012, 7th ed. § 10.5, pp. 698-699).





Hamilton Paths

Definition

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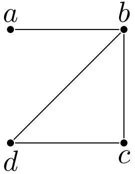
Exercise

Find a Hamilton path in the following graph:



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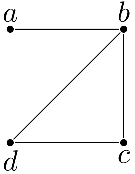

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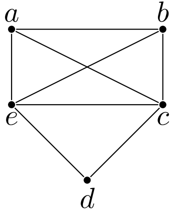

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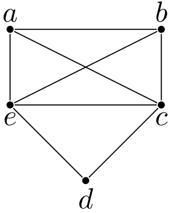

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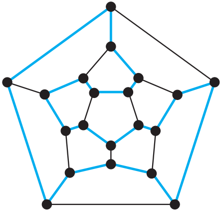
A Hamilton circuit is a, b, c, d, e, a.

Hamilton Circuits

Example

Solution to the icosian puzzle.[†]



[†]Figure source: (Rosen 2012, § 10.5, Fig. 9).

Hamilton Circuits

Theorem 3 (Dirac's Theorem, p. 548)

If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

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Theorem 4 (Ore's Theorem, p. 548)

If G is a simple graph with n vertices with $n \geq 3$ such that $\delta(u) + \delta(v) \geq n$ for every pair of non-adjacent vertices u and v in G, then G has a Hamilton circuit.

References

- Rosen, K. H. (2004). *Matemática Discreta y sus Aplicaciones*. 5th ed. Translated by José Manuel Pérez Morales and others. McGraw-Hill (cit. on p. 2).
- — (2012). Discrete Mathematics and Its Applications. 7th ed. McGraw-Hill (cit. on pp. 3, 16, 17, 23–25, 32).