CM0246 Discrete Structures
Equivalence Relations

Andrés Sicard-Ramírez

Universidad EAFIT

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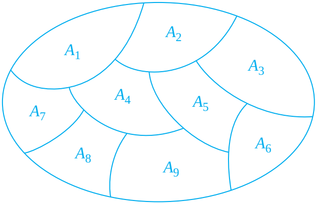
Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook (Rosen 2004).

Introduction

Equivalence relations split sets into disjoint classes of equivalent elements.[†]



[†]Figure source: (Rosen 2012, § 9.5, Fig. 1).

Equivalence Relations

Definition

A relation on a set A is an **equivalence relation** iff it is reflexive, symmetric and transitive.

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Example (words of the same length)

$$\Sigma = \{a, b, \ldots, z\},$$
$$\Sigma^* = \{\, w \mid w \text{ is a word on } \Sigma \,\},$$

$$R = \{\, (w, w') \mid l(w) = l(w') \,\} \subseteq \Sigma^* \times \Sigma^*.$$

Equivalence Relations

Exercise

Let $A = \{a, e, i, o, u\}$. Is the equality relation on A an equivalence relation?

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Example

$$\mathrm{FUN} = \{\, f \mid f : \{0,1\} \to \{0,1\} \,\},$$

$$R = \{\, (f,g) \mid f(1) = g(1) \,\} \subseteq \mathrm{FUN} \times \mathrm{FUN}.$$

Equivalence Relations

Exercise

Let A be a unitary set. It is possible to define an equivalence relation on A?

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Define an equivalence relation on a finite/infinite set.

Equivalence Relations

Definition

Let m and n be integers and let d be a positive integer. The number m **is congruent to** n **modulo** d, denoted by $m \equiv n \pmod{d}$, iff $d \mid (m - n)$.

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Example

The congruence relation is an equivalence relation.

Equivalence Classes

Definition

Let R be an equivalence relation on a set A. The **equivalence class** of $a \in A$ with respect to R is defined by

$$[a]_R = \{ s \in A \mid (a, s) \in R \}.$$

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Example (Words of the same length)

$$\begin{aligned} [\lambda] &= \{\lambda\}, \\ [a] &= \{a, b, \ldots, z\} = [k], \\ [aa] &= \{aa, ab, \ldots, az, ba, bb, \ldots, bz, \ldots za, \ldots zz\}, \\ [hgbj] &= \{\, w \mid l(w) = 4 \,\}. \end{aligned}$$

Equivalence Classes

Example (equality relation)

Whiteboard.

Equivalence Classes

Example (equality relation)

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Example (Cartesian product)

Whiteboard.

Equivalence Relations

Theorem

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Proof.

i) $aRb \Rightarrow [a] \subseteq [b]$

1	aRb.	(hypothesis)
2	Let $c \in [a]$.	
3	aRc.	(def. of $[a]$)
4	bRa.	(R is symmetric)
5	bRc.	(R is transitive)
6	$c \in [b]$.	(def. of $[b]$)
7	Therefore $[a] \subseteq [b]$.	■

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Theorem

Let R be an equivalence relation on a set A. For all $a, b \in A$,

$$aRb \Rightarrow [a] = [b].$$

Proof.

ii) $aRb \Rightarrow [b] \subseteq [a]$

1	aRb.	(hypothesis)
2	Let $c \in [b]$.	
3	bRc.	(def. of $[b]$)
4	aRc.	(R is transitive)
5	$c \in [a]$.	(def. of $[a]$)
6	Therefore $[b] \subseteq [a]$.	■

Equivalence Relations

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$$[a] = [b] \Rightarrow [a] \cap [b] \neq \emptyset.$$

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$$[a] = [b] \Rightarrow [a] \cap [b] \neq \emptyset.$$

Proof.

1	$[a] = [b]$.	(hypothesis)
2	$[a] = \{a, \ldots\}$.	(R is reflexive)
3	Therefore $[a] \cap [b] \neq \emptyset$.	■

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$$[a] \cap [b] \neq \emptyset \Rightarrow aRb.$$

Proof.

1	$[a] \cap [b] \neq \emptyset$.	(hypothesis)
2	Let c such that $c \in [a]$ and $c \in [b]$.	
3	aRc y bRc.	(def. of $[a]$ and $[b]$)
4	cRb.	(R is symmetric)
5	Therefore aRb.	(R is transitive)

■

Equivalence Relations

Theorem 1 (p. 476)

Let R be an equivalence relation on a set A. For all $a, b \in A$, the following statements are equivalent:

$$aRb, \tag{1}$$

$$[a] = [b], \tag{2}$$

$$[a] \cap [b] \neq \emptyset. \tag{3}$$

Proof.

(1) $\Rightarrow$ (2) (previous theorem),
(2) $\Rightarrow$ (3) (previous theorem) and
(3) $\Rightarrow$ (1) (previous theorem). ■

Equivalence Relations and Partitions

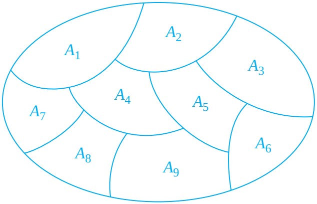
Definition

A **partition** of a set A is a collection of subsets $\{ A_i \mid i \in I \}$ of A such that:[†]

i) $A_i \neq \emptyset$, for $i \in I$,

ii) $A_i \cap A_j = \emptyset$ when $i \neq j$ (disjoint subsets) and

iii) $\bigcup_{i \in I} A_i = A$.



[†]Figure source: (Rosen 2012, § 9.5, Fig. 1).

Equivalence Relations and Partitions

Theorem 2 (p. 477)

Let R be an equivalence relation on a set A. Then the equivalence classes of R form a partition of A.

Proof.

The collection of subsets is given by

$$\left\{ A_{[a]_R} \;\middle|\; [a]_R \text{ is an equivalence class respect to } R \right\}.$$

Using the above collection, the conditions i), ii) and iii) are satisfied. ■

Equivalence Relations and Partitions

Theorem 2 (Rosen (5th ed.), p. 477)

Given a partition $\{ A_i \mid i \in I \}$ of a set A, there is an equivalence relation R that has the sets A_i as its equivalence classes.

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Example

Given a partition to build the equivalence relation associated.

Equivalence Relations and Partitions

Proof.

Let R be the relation defined by

$$R = \{ (a, b) \mid a, b \in A_i \} .$$

R is a relation of equivalence:

Equivalence Relations and Partitions

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$$R = \{ (a, b) \mid a, b \in A_i \} .$$

R is a relation of equivalence:

- Reflexivity and symmetry

  Direct from the definition of R.

Continued on next slide

Equivalence Relations and Partitions

Proof (continuation).

$$R = \{\, (a,b) \mid a,b \in A_i \,\}\,.$$

- Transitivity
  1) aRb and bRc.
  2) Exists $X \in \{\, A_i \mid i \in I \,\}$ such that $a, b \in X$ by definition of R.
  3) Exists $Y \in \{\, A_i \mid i \in I \,\}$ such that $b, c \in Y$ by definition of R.
  4) $X = Y$ because $b \in X$ and $b \in Y$ and the A_is are disjoints.
  5) aRc ($a, c \in X$ and def. of R).

Equivalence Relations and Partitions

Proof (continuation).

$$R = \{ (a,b) \mid a, b \in A_i \}.$$

- Transitivity
  1) aRb and bRc.
  2) Exists $X \in \{ A_i \mid i \in I \}$ such that $a, b \in X$ by definition of R.
  3) Exists $Y \in \{ A_i \mid i \in I \}$ such that $b, c \in Y$ by definition of R.
  4) $X = Y$ because $b \in X$ and $b \in Y$ and the A_is are disjoints.
  5) aRc ($a, c \in X$ and def. of R).

Now, $[a]_R = \{ s \mid (a,s) \in R \}$ and by the definition of the relation R, these equivalence classes correspond to the sets A_i. ■

References

Rosen, K. H. (2004). *Matemática Discreta y sus Aplicaciones*. 5th ed. Translated by José Manuel Pérez Morales and others. McGraw-Hill (cit. on p. 2).

— (2012). Discrete Mathematics and Its Applications. 7th ed. McGraw-Hill (cit. on pp. 3, 29).