CM0246 Discrete Structures
Cardinality

Andrés Sicard-Ramírez

Universidad EAFIT

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Cardinality

Definition

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Injunction, surjection or bijection?

Draw figures in the whiteboard.

Cardinality

Example

(Proofs on the whiteboard)

- $|\mathbb{Z}^+| = |\mathbb{N}|$.

Cardinality

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- $|\mathbb{Z}^+| = |\mathbb{N}|$.
- $|\mathbb{N}| = |\mathrm{Even}|$, where $\mathrm{Even} = \{\, 2n \mid n \in \mathbb{N} \,\}$.

Cardinality

Example

(Proofs on the whiteboard)

- $|\mathbb{Z}^+| = |\mathbb{N}|$.
- $|\mathbb{N}| = |\mathrm{Even}|$, where $\mathrm{Even} = \{\, 2n \mid n \in \mathbb{N} \,\}$.
- $|\mathbb{N}| = |M_k|$, where M_k is the set of the non-negative multiples of $k \in \mathbb{Z}^+$, i.e. $M_k = \{\, nk \mid n \in \mathbb{N} \,\}$.

Cardinality

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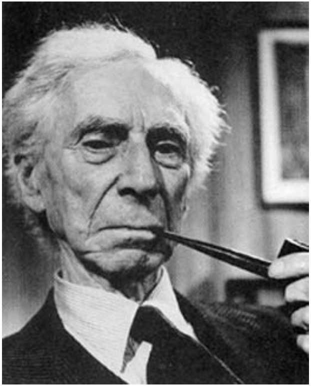
- $|\mathbb{N}| = |\mathrm{Even}|$, where $\mathrm{Even} = \{\, 2n \mid n \in \mathbb{N} \,\}$.
- $|\mathbb{N}| = |M_k|$, where M_k is the set of the non-negative multiples of $k \in \mathbb{Z}^+$, i.e. $M_k = \{\, nk \mid n \in \mathbb{N} \,\}$.
- $|[0,1]| = |[a,b]|$, where $a, b \in \mathbb{R}$ and $a < b$.

Cardinality



(1872 – 1970)†

'The possibility that whole and part may have the same number of terms is, it must be confessed, shocking to common-sense.' (Russell 1903, p. 358)

†Image from the MacTutor History of Mathematics Archive.

Cardinality

Example (Lipschutz (1998), Solved problem 6.2, p. 153)

Show that $|[0, 1]| = |(0, 1)|$.

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Solution

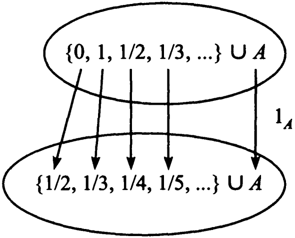
Note that

$$\begin{aligned} [0,1] &= \{0, 1, 1/2, 1/3, 1/4, \ldots\} \cup A \\ (0,1) &= \{1/2, 1/3, 1/4, \ldots\} \cup A \end{aligned}$$

where

$$\begin{aligned} A &= [0,1] - \{0, 1, 1/2, 1/3, 1/4, \ldots\} \\ &= (0,1) - \{1/2, 1/3, 1/4, \ldots\}. \end{aligned}$$

Solution (continuation)



From the figure† we define the bijective function $f : [0, 1] \to (0, 1)$ by

$$f(x) = \begin{cases} 1/2, & \text{if } x = 0; \\ 1/(n+1), & \text{if } x = 1/n \text{ where } n \in \mathbb{Z}^+; \\ x, & \text{otherwise.} \end{cases}$$

†Figure source: (Lipschutz 1998, Fig. 6.5).

Cardinality

Exercise

Let A and B be sets. Show $|A \times B| = |B \times A|$.

Enumerable and Non-Enumerable Sets

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Has all the infinite sets the same cardinality?

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Enumerable and Non-Enumerable Sets

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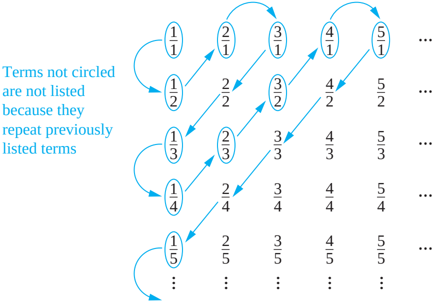
Example

Whiteboard.

Enumerable and Non-Enumerable Sets

Example (the positive rational numbers are enumerable[†])



Remark: We do not define explicitly the function, but a method (program) for enumerating the set.

[†]Figure source: (Rosen 2012, § 2.5, Fig. 3).

Enumerable and Non-Enumerable Sets

Theorem

The interval $(0, 1)$ is non-enumerable.

Proved on next slide

Enumerable and Non-Enumerable Sets

Proof.

Let's suppose $(0, 1)$ is enumerable.

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14}\ldots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24}\ldots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34}\ldots \\ &\vdots \end{aligned}$$

Enumerable and Non-Enumerable Sets

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Let $r = 0.d_1d_2d_3\ldots \in (0, 1)$, where

$$d_i = \begin{cases} 4, & \text{if } d_{ii} \neq 4; \\ 5, & \text{if } d_{ii} = 4. \end{cases}$$

Enumerable and Non-Enumerable Sets

Proof.

Let's suppose $(0,1)$ is enumerable.

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14}\dots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24}\dots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34}\dots \\ &\vdots \end{aligned}$$

Let $r = 0.d_1d_2d_3\ldots \in (0,1)$, where

$$d_i = \begin{cases} 4, & \text{if } d_{ii} \neq 4; \\ 5, & \text{if } d_{ii} = 4. \end{cases}$$

The number r does not belong to the above enumeration. Therefore $(0,1)$ is non-enumerable. ■

Enumerable and Non-Enumerable Sets

Theorem

Let A and B be sets such that $A \subseteq B$. If A is non-enumerable then B is non-enumerable.

Enumerable and Non-Enumerable Sets

Theorem

The set of the real numbers is non-enumerable.

Enumerable and Non-Enumerable Sets

Theorem

The set of the real numbers is non-enumerable.

Proof.

The interval $(0, 1)$ is a non-enumerable subset of $\mathbb{R}$. Therefore (using a previous theorem), $\mathbb{R}$ is non-enumerable. ■

Enumerable and Non-Enumerable Sets

Theorem

The set of the real numbers is non-enumerable.

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Remark

Comment about the continuum hypothesis.

Enumerable and Non-Enumerable Sets

Remark

The quadratic formulae are the solution to the quadratic equation

$$ax^2 + bx + c = 0.$$

Two quadratic formulae are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and

$$x = \frac{-2c}{b \mp \sqrt{b^2 - 4ac}}. \tag{1}$$

Enumerable and Non-Enumerable Sets

Example

Show that $|(-1,1)| = |\mathbb{R}|$.

Enumerable and Non-Enumerable Sets

Example

Show that $|(-1, 1)| = |\mathbb{R}|$.

Solution

The function $f : (-1, 1) \to \mathbb{R}$ defined by

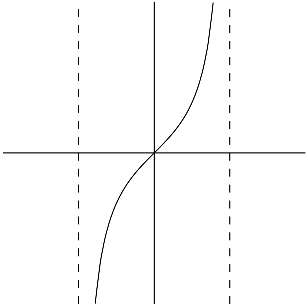
$$f(x) = \frac{x}{1 - x^2}$$

has as inverse the function $f^{-1} : \mathbb{R} \to (-1, 1)$ given by (obtained using the quadratic formula (1))

$$f^{-1}(x) = \frac{2x}{1 + \sqrt{1 + 4x^2}}.$$

Enumerable and Non-Enumerable Sets

Solution (continuation)



$$f(x) = \frac{x}{1 - x^2}$$

Since the function f is a bijection then $|(-1, 1)| = |\mathbb{R}|$. Source: Munkres (2000, Example § 18.5).

References

- Lipschutz, S. (1998). Schaum's Outline of Theory and Problems of Set Theory and Related Topics. 2nd ed. McGraw-Hill (cit. on pp. 10–12).
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