

## Section 1.2 Propositional Equivalences

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A *tautology* is a proposition which is always true.

Classic Example:  $P \vee \neg P$

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A *contradiction* is a proposition which is always false.

Classic Example:  $P \wedge \neg P$

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A *contingency* is a proposition which neither a tautology nor a contradiction.

Example:  $(P \wedge Q) \rightarrow R$

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Two propositions  $P$  and  $Q$  are *logically equivalent* if  $P \leftrightarrow Q$  is a tautology. We write

$$P \equiv Q$$


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Example:  $(P \wedge Q) \equiv (Q \wedge P) \equiv (P \rightarrow Q)$

Proof:

The left side and the right side must have the same truth values independent of the truth value of the component propositions.

To show a proposition is not a tautology: use an *abbreviated* truth table

- try to find a *counter example* or to *disprove* the assertion.
- search for a case where the proposition is false

Case 1: Try left side false, right side true

Left side false: only one of  $P \vee Q$  or  $Q \wedge P$  need be false.

1a. Assume  $P \vee Q = F$ .

Then  $P = T$ ,  $Q = F$ . But then right side  $Q \wedge P = F$ . Oops, wrong guess.

1b. Try  $Q \wedge P = F$ . Then  $Q = T$ ,  $P = F$ . Then  $P \vee Q = F$ . Another wrong guess.

Case 2. Try left side true, right side false

If right side is false, P and Q cannot have the same truth value.

2a. Assume  $P = T, Q = F$ .  
Then  $P \wedge Q = F$  and the conjunction must be false so the left side cannot be true in this case. Another wrong guess.

2b. Assume  $Q = T, P = F$ .  
Again the left side cannot be true.

We have exhausted all possibilities and not found a counterexample. The two propositions must be logically equivalent.

Note: Because of this equivalence, *if and only if* or *iff* is also stated as *is a necessary and sufficient condition for*.

Some famous logical equivalences:

<b>Logical Equivalences</b>					
$P$	$T$	$P$			Identity
$P$	$F$	$P$			Domination
$P$	$T$	$T$			
$P$	$F$	$F$			Idempotency
$P$	$P$	$P$			
$P$	$P$	$P$			Double negation
$\neg(\neg P)$		$P$			
$P$	$Q$	$Q$	$P$		Commutativity
$P$	$Q$	$Q$	$P$	$(Q \ R)$	Associativity
$(P \ Q)$	$R$	$P$	$(Q \ R)$		
$(P \ Q)$	$R$	$P$	$(Q \ R)$		

$P \ (Q \ R)$	Distributivity
$(P \ Q) \ (P \ R)$	
$P \ (Q \ R)$	
$(P \ Q) \ (P \ R)$	
$\neg(P \ Q) \ \neg P \ \neg Q$	DeMorgan's laws
$\neg(P \ Q) \ \neg P \ \neg Q$	
$P \ Q \ \neg P \ Q$	Implication
$P \ \neg P \ T$	Tautology
$P \ \neg P \ F$	Contradiction
$P \ T \ P$	
$P \ F \ P$	
$(P \ Q) \ (Q \ P)$	Equivalence
$(P \ Q)$	
$(P \ Q) \ (P \ \neg Q)$	Absurdity
$\neg P$	
$(P \ Q) \ (\neg Q \ \neg P)$	Contrapositive
$P \ (P \ Q) \ P$	Absorption
$P \ (P \ Q) \ P$	
$(P \ Q) \ R$	Exportation
$P \ (Q \ R)$	

Note: equivalent expressions can always be substituted for each other in a more complex expression - useful for simplification.

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## ***Normal or Canonical Forms***

Unique representations of a proposition

Examples:

Construct a simple proposition of two variables which is true only when

- P is true and Q is false:

$$P \wedge \neg Q$$

- P is true and Q is true:

$$P \wedge Q$$

- P is true and Q is false or P is true and Q is true:

$$(P \wedge \neg Q) \vee (P \wedge Q)$$


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A disjunction of conjunctions where

- every variable or its negation is represented once in each conjunction (a *minterm*)

- each minterms appears only once

### ***Disjunctive Normal Form (DNF)***

Important in switching theory, simplification in the design of circuits.

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Method: To find the minterms of the DNF.

- Use the rows of the truth table where the proposition is 1 or True
- If a zero appears under a variable, use the negation of the propositional variable in the minterm
- If a one appears, use the propositional variable.

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Example:

Find the DNF of  $(P \vee Q) \wedge \neg R$

P	Q	R	$(P \vee Q) \wedge \neg R$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

There are 5 cases where the proposition is true, hence 5 minterms. Rows 1,2,3, 5 and 7 produce the following disjunction of minterms:

$$(P \wedge Q) \wedge \neg R$$

$$\begin{aligned} & (\neg P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \\ & \vee (P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge \neg R) \end{aligned}$$

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Note that you get a *Conjunctive Normal Form* (CNF) if you negate a DNF and use DeMorgan's Laws.

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