

Category Theory and Functional Programming

Natural Transformations

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, pages, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Abramsky and Tzevelekos 2011].

Outline

Introduction

Definition of a Natural Transformation

Examples of Natural Transformations

Natural Isomorphisms

Natural Transformations Between Hom-Functors

Compositions of Natural Transformations

Functor Category

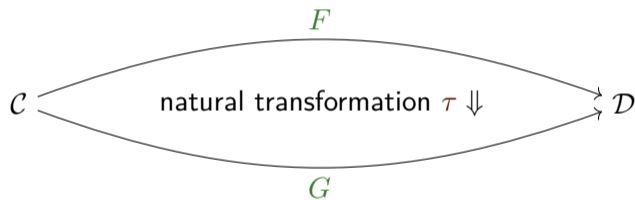
References

Introduction

Introduction

Description

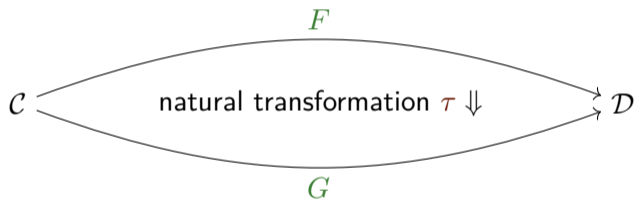
A natural transformation is a structure preserving mapping (i.e. preserves composition of arrows and identity arrows) between 'parallel' functors.



Introduction

Description

A natural transformation is a structure preserving mapping (i.e. preserves composition of arrows and identity arrows) between 'parallel' functors.



'As Eilenberg-Mac Lane first observed, "category" has been defined in order to be able to define "functor" and "functor" has been defined in order to be able to define "natural transformation"'. [Mac Lane 1998, p. 48]

Definition of a Natural Transformation

Definition of a Natural Transformation

Definition

Let \mathcal{C} and \mathcal{D} be categories and let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be functors. A **natural transformation**[†]

$$\tau : F \Rightarrow G$$

is a family of arrows in \mathcal{D} indexed by objects A of \mathcal{C} ,

$$\{ \tau_A : F_0 A \rightarrow G_0 A \}_{A \in \text{Obj}(\mathcal{C})} \quad \text{(components of } \tau \text{ at } A)$$

such that, for all $f : A \rightarrow B$ in \mathcal{C} ,

$$(G_1 f) \circ \tau_A = \tau_B \circ (F_1 f) \quad \text{(naturality condition)}$$

(continued on next slide)

[†]The textbook uses the notation $\tau : F \rightarrow G$.

Definition of a Natural Transformation

Definition (continuation)

That is, the following diagram commutes.

$$\begin{array}{ccccc} A & & F_0 A & \xrightarrow{\tau_A} & G_0 A \\ \downarrow f & & \downarrow F_1 f & & \downarrow G_1 f \\ B & & F_0 B & \xrightarrow{\tau_B} & G_0 B \end{array}$$

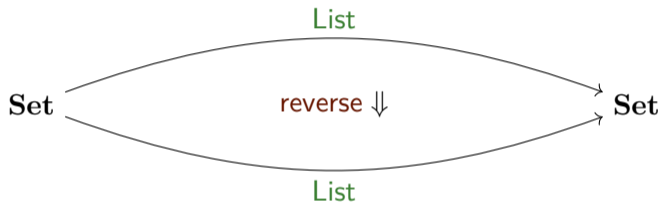
$$\left((G_1 f) \circ \tau_A = \tau_B \circ (F_1 f) \right)$$

Examples of Natural Transformations

Examples of Natural Transformations

Example

We shall define the natural transformation `reverse` on the functor `List : Set → Set`.



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Examples of Natural Transformations

Example (continuation)

$\text{reverse} : \text{List} \Rightarrow \text{List}$

$\text{reverse}_X : \text{List}_0 X \rightarrow \text{List}_0 X$

$\text{reverse}_X : [X] \rightarrow [X]$

$\text{reverse}_X [x_1, \dots, x_n] := [x_n, \dots, x_1]$

Examples of Natural Transformations

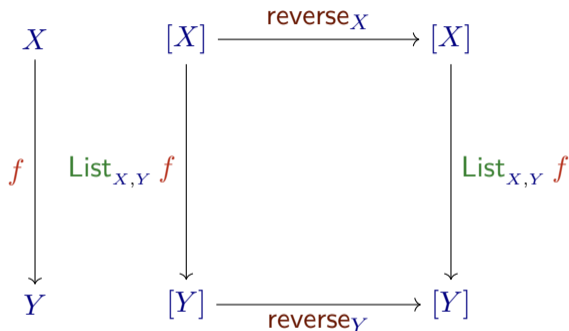
Example (continuation)

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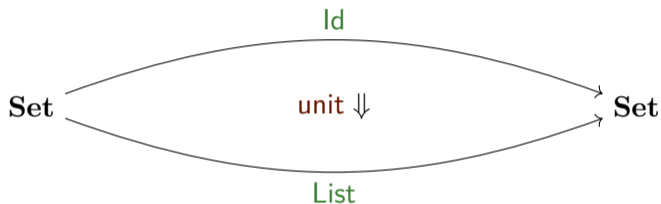


For each $f : X \rightarrow Y$ in \mathbf{Set} , the above diagram commutes by naturality.

Examples of Natural Transformations

Example

We shall define the natural transformation **unit** on the functors **Id**, **List** : **Set** \rightarrow **Set**.



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Examples of Natural Transformations

Example (continuation)

$\text{unit} : \text{Id} \Rightarrow \text{List}$

$\text{unit}_X : \text{Id}_0 X \rightarrow \text{List}_0 X$

$\text{unit}_X : X \rightarrow [X]$

$\text{unit}_X x := [x]$

Examples of Natural Transformations

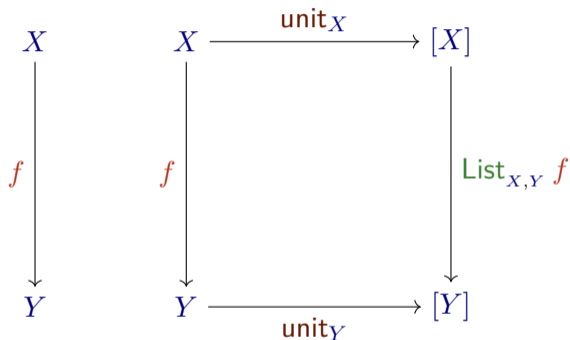
Example (continuation)

$\text{unit} : \text{Id} \Rightarrow \text{List}$

$\text{unit}_X : \text{Id}_0 X \rightarrow \text{List}_0 X$

$\text{unit}_X : X \rightarrow [X]$

$\text{unit}_X x := [x]$

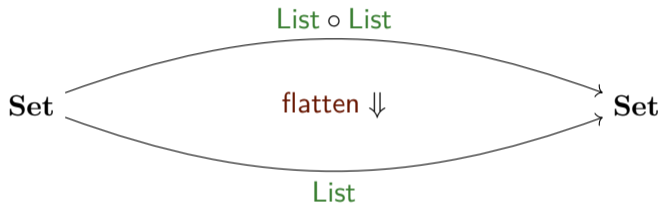


For each $f : X \rightarrow Y$ in \mathbf{Set} , the above diagram commutes by naturality.

Examples of Natural Transformations

Example

We define the natural transformation **flatten** on the functors $\mathbf{List} \circ \mathbf{List}, \mathbf{List} : \mathbf{Set} \rightarrow \mathbf{Set}$.



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Examples of Natural Transformations

Example (continuation)

$\text{flatten} : \text{List} \circ \text{List} \Rightarrow \text{List}$

$\text{flatten}_X : \text{List}_0 (\text{List}_0 X) \rightarrow \text{List}_0 X$

$\text{flatten}_X : [[X]] \rightarrow [X]$

$\text{flatten}_X [[x_1^1, \dots, x_{n_1}^1], \dots, [x_1^k, \dots, x_{n_k}^k]] := [x_1^1, \dots, x_{n_1}^1, \dots, x_1^k, \dots, x_{n_k}^k]$

Examples of Natural Transformations

Example (continuation)

$\text{flatten} : \text{List} \circ \text{List} \Rightarrow \text{List}$

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For each $f : X \rightarrow Y$ in **Set**, the following diagram commutes by naturality.

$$\begin{array}{ccccc} X & & [[X]] & \xrightarrow{\text{flatten}_X} & [X] \\ \downarrow f & & \downarrow (\text{List} \circ \text{List})_{X,Y} f & & \downarrow \text{List}_{X,Y} f \\ Y & & [[Y]] & \xrightarrow{\text{flatten}_Y} & [Y] \end{array}$$

Examples of Natural Transformations

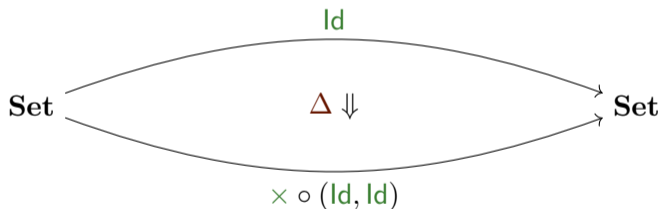
Exercise 1

Verify naturality of `reverse`, `unit` and `flatten`.

Examples of Natural Transformations

Example

Let Id be the identity functor in \mathbf{Set} and let $\times \circ (\text{Id}, \text{Id}) : \mathbf{Set} \rightarrow \mathbf{Set}$ the functor sending every set X to $X \times X$ and every function f to $f \times f$. We shall define the natural transformation **diagonal** Δ .



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Examples of Natural Transformations

Example (continuation)

$$\Delta : \text{Id} \Rightarrow \times \circ (\text{Id}, \text{Id})$$

$$\Delta_X : \text{Id}_0 X \rightarrow (\times \circ (\text{Id}, \text{Id}))_0 X$$

$$\Delta_X : X \rightarrow X \times X$$

$$\Delta_X x := (x, x)$$

Examples of Natural Transformations

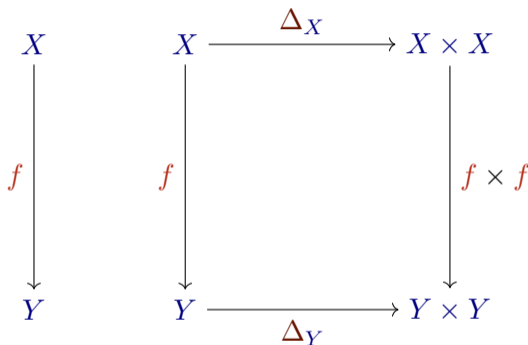
Example (continuation)

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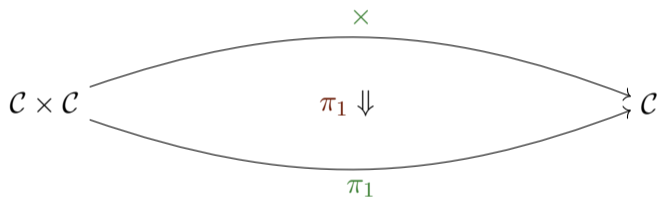


For each $f : X \rightarrow Y$ in **Set**, the above diagram commutes by naturality.

Examples of Natural Transformations

Example

Let $\times, \pi_1 : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ be the product and first projection functors, respectively. We shall define the natural transformation **first projection** π_1 .



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Examples of Natural Transformations

Example (continuation)

$$\pi_1 : \times \Rightarrow \pi_1$$

$$\pi_{1(A,B)} : \times_0(A, B) \rightarrow (\pi_1)_0(A, B)$$

$$\pi_{1(A,B)} : A \times B \rightarrow A$$

Examples of Natural Transformations

Example (continuation)

$$\pi_1 : \times \Rightarrow \pi_1$$

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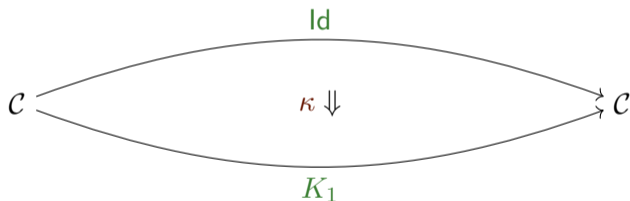
$$\begin{array}{ccccc} (A, B) & A \times B & \xrightarrow{\pi_{1(A,B)}} & A & \\ \downarrow (f, g) & \downarrow f \times g & & \downarrow f & \\ (A', B') & A' \times B' & \xrightarrow{\pi_{1(A',B')}} & A' & \end{array}$$

For each $(f, g) : (A, B) \rightarrow (A', B')$ in $\mathcal{C} \times \mathcal{C}$, the above diagram commutes by naturality.

Examples of Natural Transformations

Example

Let \mathcal{C} be a category with terminal object 1 , let Id be the identity functor in \mathcal{C} and let $K_1 : \mathcal{C} \rightarrow \mathcal{C}$ be the functor mapping all objects of \mathcal{C} to 1 and all arrows of \mathcal{C} to id_1 . We shall define the natural transformation κ .



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Examples of Natural Transformations

Example (continuation)

$$\kappa : \text{Id} \Rightarrow K_1$$

$$\kappa_A : \text{Id}_0 A \rightarrow (K_1)_0 A$$

$$\kappa_A : A \rightarrow 1$$

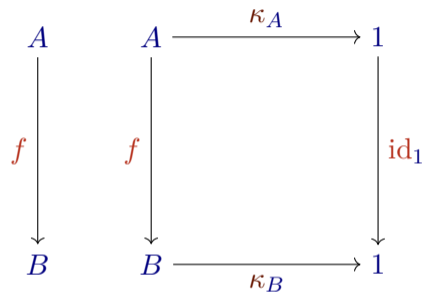
Examples of Natural Transformations

Example (continuation)

$$\kappa : \text{Id} \Rightarrow K_1$$

$$\kappa_A : \text{Id}_0 A \rightarrow (K_1)_0 A$$

$$\kappa_A : A \rightarrow 1$$



For each $f : A \rightarrow B$ in \mathcal{C} , the above diagram commutes by naturality.

Exercises

Exercise 2

Verify naturality of the natural transformation κ .

Natural Isomorphisms

Natural Isomorphisms

Definition

A natural transformation

$$\begin{aligned}\tau &: F \Rightarrow G \\ \tau_A &: F_0 A \rightarrow G_0 A, \quad \text{for all } A \text{ in } \mathcal{C}\end{aligned}$$

is a **natural isomorphism** iff each τ_A is an isomorphism.

Natural Isomorphisms

Definition

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Notation

A natural isomorphism is denoted by $\tau : F \xrightarrow{\cong} G$.

Natural Isomorphisms

Example

Let \mathcal{C} be a category with products and let the functors (textbook and [Awodey 2010, § 7.4])

$$-_1 \times (-_2 \times -_3) : \mathcal{C}^3 \rightarrow \mathcal{C} \quad \text{and} \quad (-_1 \times -_2) \times -_3 : \mathcal{C}^3 \rightarrow \mathcal{C}.$$

The natural isomorphism a shows that the product is associative.

$$\begin{aligned} a &: -_1 \times (-_2 \times -_3) \xrightarrow{\cong} (-_1 \times -_2) \times -_3 \\ a_{A,B,C} &: A \times (B \times C) \xrightarrow{\cong} (A \times B) \times C \\ a_{A,B,C} &:= \langle \langle \pi_1, \pi_1 \circ \pi_2 \rangle, \pi_2 \circ \pi_2 \rangle. \end{aligned}$$

Natural Isomorphisms

Example

Let \mathcal{C} be a category with binary products. We define the functor $\bar{\times} : \mathcal{C}^2 \rightarrow \mathcal{C}$ by (textbook and [Awodey 2010, Example 7.8])

$$\bar{\times}_0(A, B) := \times_0(B, A) = B \times A, \quad \bar{\times}_1(f, g) := \times_1(g, f) = g \times f.$$

The natural isomorphism s shows that the product is symmetric.

$$\begin{aligned} s : \times &\xrightarrow{\cong} \bar{\times} \\ s_{A,B} : A \times B &\xrightarrow{\cong} B \times A \\ s_{A,B} &:= \langle \pi_2, \pi_1 \rangle. \end{aligned}$$

$$\begin{array}{ccc} (A, B) & & A \times B \xrightarrow{s_{A,B}} B \times A \\ \downarrow (f, g) & & \downarrow f \times g \quad \downarrow g \times f \\ (A', B') & & A' \times B' \xrightarrow{s_{A',B'}} B' \times A' \end{array}$$

Natural Isomorphisms

Example

Let \mathcal{C} be a category with binary products and terminal object 1 and let the functors

$$1 \times - : \mathcal{C} \rightarrow \mathcal{C}$$

$$(1 \times -)_0 A := 1 \times A$$

$$(1 \times -)_1 f := \langle \text{id}_1, f \rangle,$$

$$- \times 1 : \mathcal{C} \rightarrow \mathcal{C}$$

$$(- \times 1)_0 A := A \times 1$$

$$(- \times 1)_1 f := \langle f, \text{id}_1 \rangle.$$

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Natural Isomorphisms

Example (continuation)

The natural isomorphisms l and r show that 1 is the unit of the product.

$$\begin{aligned}l &: 1 \times - \xrightarrow{\cong} \text{Id} \\l_A &: 1 \times A \xrightarrow{\cong} A \\l_A &:= \pi_2,\end{aligned}$$

$$\begin{aligned}r &: - \times 1 \xrightarrow{\cong} \text{Id} \\r_A &: A \times 1 \xrightarrow{\cong} A \\r_A &:= \pi_1.\end{aligned}$$

$$\begin{array}{ccc}A & 1 \times A & \xrightarrow{l_A} & A \\f \downarrow & \langle \text{id}_1, f \rangle \downarrow & & \downarrow f \\B & 1 \times B & \xrightarrow{l_B} & B\end{array}$$

$$\begin{array}{ccc}A & A \times 1 & \xrightarrow{r_A} & A \\f \downarrow & \langle f, \text{id}_1 \rangle \downarrow & & \downarrow f \\B & B \times 1 & \xrightarrow{r_B} & B\end{array}$$

Natural Isomorphisms

Exercise 3

Verify that the families of arrows $s_{A,B}$, l_A and r_A , from the previous examples, are natural isomorphisms (textbook, Exercise 53).

Natural Isomorphisms

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Verify that the families of arrows $s_{A,B}$, l_A and r_A , from the previous examples, are natural isomorphisms (textbook, Exercise 53).

Remark

Because natural isomorphisms are a self-dual notion, we get also natural isomorphisms s , l and r for a category with binary coproducts and initial object.

Natural Transformations Between Hom-Functors

Natural Transformations Between Hom-Functors

Definition

Let \mathcal{C} be a (locally small) category and let A and B be objects of \mathcal{C} . Recall the hom-functors

$$\mathcal{C}(A, -) : \mathcal{C} \rightarrow \mathbf{Set} \quad (\text{covariant})$$

$$\mathcal{C}(-, B) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set} \quad (\text{contravariant})$$

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Natural Transformations Between Hom-Functors

Definition (continuation, first notation)

Let $f : A \rightarrow B$ in \mathcal{C} . We define the natural transformation $\mathcal{C}(f, -)$ and we show its naturality condition for $h : C \rightarrow D$ in \mathcal{C} .

$$\mathcal{C}(f, -) : \mathcal{C}(B, -) \Rightarrow \mathcal{C}(A, -)$$

$$\mathcal{C}(f, -)_C : \mathcal{C}(B, C) \rightarrow \mathcal{C}(A, C)$$

$$\mathcal{C}(f, -)_C g : \mathcal{C}(A, C)$$

$$\mathcal{C}(f, -)_C g := g \circ f$$

$$\begin{array}{ccc} \mathcal{C} & & \mathcal{C}(B, C) \xrightarrow{\mathcal{C}(f, -)_C} \mathcal{C}(A, C) \\ \downarrow h & & \downarrow \mathcal{C}(B, -)_1 h \quad \downarrow \mathcal{C}(A, -)_1 h \\ D & & \mathcal{C}(B, D) \xrightarrow{\mathcal{C}(f, -)_D} \mathcal{C}(A, D) \end{array}$$

where $\mathcal{C}(f, -)_C = \mathcal{C}(-, C)_0 f$.

Natural Transformations Between Hom-Functors

Definition (continuation, second notation)

Let $f : A \rightarrow B$ in \mathcal{C} . We define the natural transformation $\mathcal{C}(f, -)$ and we show its naturality condition for $h : C \rightarrow D$ in \mathcal{C} .

$$\begin{array}{ccc} \mathcal{C}(f, -) : \mathcal{C}(B, -) \Rightarrow \mathcal{C}(A, -) & & \\ \mathcal{C}(f, -)_C : \mathcal{C}(B, C) \rightarrow \mathcal{C}(A, C) & & \\ \mathcal{C}(f, -)_C g : \mathcal{C}(A, C) & & \\ \mathcal{C}(f, -)_C g := g \circ f & & \end{array} \quad \begin{array}{ccc} C & \mathcal{C}(B, C) & \xrightarrow{\mathcal{C}(f, -)_C = \mathcal{C}(f, C)} & \mathcal{C}(A, C) \\ \downarrow h & \downarrow \mathcal{C}(B, h) & & \downarrow \mathcal{C}(A, h) \\ D & \mathcal{C}(B, D) & \xrightarrow{\mathcal{C}(f, -)_C = \mathcal{C}(f, D)} & \mathcal{C}(A, D) \end{array}$$

Natural Transformations Between Hom-Functors

Exercise 4

Define the natural transformation $\mathcal{C}(-, f) : \mathcal{C}(-, A) \Rightarrow \mathcal{C}(-, B)$ and verify its naturality (text-book, Exercise 55).

Yoneda Lemma

Yoneda embedding

Let \mathcal{C} be a locally small category and let A and B be objects of \mathcal{C} . For each natural transformation between hom-functors $\tau : \mathcal{C}(A, -) \rightarrow \mathcal{C}(B, -)$, there is a unique arrow $f : B \rightarrow A$ such that

$$\tau = \mathcal{C}(f, -).$$

Yoneda Lemma

Yoneda embedding

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$$\tau = \mathcal{C}(f, -).$$

Remark

The Yoneda lemma is a generalisation of the Yoneda embedding.

Compositions of Natural Transformations

Vertical Composition

Definition

Let \mathcal{C} be a category, let $F, G, H : \mathcal{C} \rightarrow \mathcal{D}$ be functors, and let $\tau : F \Rightarrow G$ and $\mu : G \Rightarrow H$ be natural transformations. The **vertical composition** of μ and τ is the natural transformation $\mu \circ \tau$ defined by

$$\begin{aligned} \mu \circ \tau &: F \Rightarrow H \\ (\mu \circ \tau)_A &: F_0 A \rightarrow H_0 A := \mu_A \circ \tau_A, \quad \text{for all } A \text{ in } \mathcal{C}. \end{aligned}$$

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Vertical Composition

Definition (continuation)

That is, for all $f : A \rightarrow B$ in \mathcal{C} the following diagram commutes.

$$\begin{array}{ccc} A & & F_0 A \xrightarrow{F_1 f} F_0 B \\ \downarrow f & & \downarrow \tau_A \qquad \downarrow \tau_B \\ B & & G_0 A \xrightarrow{G_1 f} G_0 B \\ & & \downarrow \mu_A \qquad \downarrow \mu_B \\ & & H_0 A \xrightarrow{H_1 f} H_0 B \end{array}$$

$$\left(\begin{array}{l} (G_1 f) \circ \tau_A = \tau_B \circ (F_1 f) \\ (H_1 f) \circ \mu_A = \mu_B \circ (G_1 f) \\ (H_1 f) \circ (\mu \circ \tau)_A = (\mu \circ \tau)_B \circ (F_1 f) \end{array} \right)$$

Vertical Composition

Exercise 5

Let \mathcal{C} be a category, let $F, G, H : \mathcal{C} \rightarrow \mathcal{D}$ be functors, and let $\tau : F \Rightarrow G$ and $\mu : G \Rightarrow H$ be natural transformations. Show that $\mu \circ \tau : F \Rightarrow H$ is a natural transformation.

Functor Category

Functor Category

Introduction

Since we can define an associative composition of natural transformations and this composition has an identity natural transformation, we can define a category where the objects are functors and the arrows are natural transformations.

Functor Category

Definition

Let \mathcal{C} be a small category and let \mathcal{D} be an arbitrary category. The **functor category** $\text{Func}(\mathcal{C}, \mathcal{D})$ is defined by

- (i) Objects: Functors $F : \mathcal{C} \rightarrow \mathcal{D}$.
- (ii) Arrows: Natural transformations $\tau : F \Rightarrow G$.

Functor Category

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- (i) Objects: Functors $F : \mathcal{C} \rightarrow \mathcal{D}$.
- (ii) Arrows: Natural transformations $\tau : F \Rightarrow G$.
- (iii) Composition of arrows: Vertical composition of natural transformations.

Functor Category

Definition

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- (i) Objects: Functors $F : \mathcal{C} \rightarrow \mathcal{D}$.
- (ii) Arrows: Natural transformations $\tau : F \Rightarrow G$.
- (iii) Composition of arrows: Vertical composition of natural transformations.
- (iv) Identity arrow

$$\text{id}_F : F \rightarrow F$$

$$(\text{id}_F)_A : F_0 A \rightarrow F_0 A \quad \text{for all } A \text{ in } \mathcal{C},$$

$$(\text{id}_F)_A := \text{id}_{(F_0 A)}.$$

References

References



Abramsky, S. and Tzevelekos, N. (2011). Introduction to Categories and Categorical Logic. In: New Structures for Physics. Ed. by Coecke, B. Vol. 813. Lecture Notes in Physics. Springer, pp. 3–94. DOI: [10.1007/978-3-642-12821-9_1](https://doi.org/10.1007/978-3-642-12821-9_1) (cit. on p. 2).



Awodey, S. [2006] (2010). Category Theory. 2nd ed. Vol. 52. Oxford Logic Guides. Oxford University Press (cit. on pp. 34, 35).



Mac Lane, S. [1971] (1998). Categories for the Working Mathematician. 2nd ed. Springer (cit. on pp. 5, 6).