Category Theory and Functional Programming
Natural Transformations

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, pages, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Abramsky and Tzevelekos 2011].

Outline

Introduction

Introduction

Description

A natural transformation is a structure preserving mapping (i.e. preserves composition of arrows and identity arrows) between 'parallel' functors.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\text{natural transformation } \tau \Downarrow"] & \mathcal{D}
\end{tikzcd}
$$

Introduction

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A natural transformation is a structure preserving mapping (i.e. preserves composition of arrows and identity arrows) between 'parallel' functors.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\text{natural transformation } \tau \Downarrow"] & & \mathcal{D} \end{tikzcd}$$

'As Eilenberg-Mac Lane first observed, "category" has been defined in order to be able to define "functor" and "functor" has been defined in order to be able to define "natural transformation".' [Mac Lane 1998, p. 48]

Definition of a Natural Transformation

Definition of a Natural Transformation

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation**†

$$\tau : F \Rightarrow G$$

is a family of arrows in $\mathcal{D}$ indexed by objects A of $\mathcal{C}$,

$$\{\, \tau_A : F_0\, A \to G_0\, A \,\}_{A \in \mathrm{Obj}(\mathcal{C})} \qquad \textbf{(components of } \tau \textbf{ at } A\textbf{)}$$

such that, for all $f : A \to B$ in $\mathcal{C}$,

$$(G_1\, f) \circ \tau_A = \tau_B \circ (F_1\, f) \qquad \text{(}\textbf{naturality}\text{ condition)}$$

(continued on next slide)

†The textbook uses the notation $\tau : F \to G$.

Definition of a Natural Transformation

Definition (continuation)

That is, the following diagram commutes.

$$\begin{tikzcd} A \arrow[d, "f"'] & F_0\,A \arrow[r, "\tau_A"] \arrow[d, "F_1\,f"'] & G_0\,A \arrow[d, "G_1\,f"] \\ B & F_0\,B \arrow[r, "\tau_B"'] & G_0\,B \end{tikzcd}$$

$$\big((G_1\,f) \circ \tau_A = \tau_B \circ (F_1\,f)\big)$$

Examples of Natural Transformations

Examples of Natural Transformations

Example

We shall define the natural transformation reverse on the functor List : $\mathbf{Set} \to \mathbf{Set}$.

$$\begin{tikzcd}[column sep=huge]
\mathbf{Set} \arrow[r, bend left, "\mathsf{List}"] \arrow[r, bend right, "\mathsf{List}"'] \arrow[r, phantom, "\mathsf{reverse}\ \Downarrow"] & \mathbf{Set}
\end{tikzcd}$$

(continued on next slide)

Examples of Natural Transformations

Example (continuation)

$$\mathsf{reverse} : \mathsf{List} \Rightarrow \mathsf{List}$$

$$\mathsf{reverse}_X : \mathsf{List}_0\, X \to \mathsf{List}_0\, X$$

$$\mathsf{reverse}_X : [X] \to [X]$$

$$\mathsf{reverse}_X\, [x_1, \ldots, x_n] := [x_n, \ldots, x_1]$$

Examples of Natural Transformations

Example (continuation)

$\mathsf{reverse} : \mathsf{List} \Rightarrow \mathsf{List}$

$\mathsf{reverse}_X : \mathsf{List}_0\, X \to \mathsf{List}_0\, X$

$\mathsf{reverse}_X : [X] \to [X]$

$\mathsf{reverse}_X\, [x_1, \ldots, x_n] := [x_n, \ldots, x_1]$

$$\begin{tikzcd} X \arrow[d, "f"'] & {[X]} \arrow[r, "\mathsf{reverse}_X"] \arrow[d, "\mathsf{List}_{X,Y}\, f"'] & {[X]} \arrow[d, "\mathsf{List}_{X,Y}\, f"] \\ Y & {[Y]} \arrow[r, "\mathsf{reverse}_Y"'] & {[Y]} \end{tikzcd}$$

For each $f : X \to Y$ in $\mathbf{Set}$, the above diagram commutes by naturality.

Examples of Natural Transformations

Example

We shall define the natural transformation unit on the functors Id, List : $\mathbf{Set} \to \mathbf{Set}$.

$$
\begin{tikzcd}
\mathbf{Set} \arrow[rr, bend left, "\mathsf{Id}"] \arrow[rr, bend right, "\mathsf{List}"'] \arrow[rr, phantom, "\mathsf{unit}\Downarrow"] & & \mathbf{Set}
\end{tikzcd}
$$

(continued on next slide)

Examples of Natural Transformations

Example (continuation)

$$\mathsf{unit} : \mathsf{Id} \Rightarrow \mathsf{List}$$

$$\mathsf{unit}_X : \mathsf{Id}_0\, X \to \mathsf{List}_0\, X$$

$$\mathsf{unit}_X : X \to [X]$$

$$\mathsf{unit}_X\, x := [x]$$

Examples of Natural Transformations

Example (continuation)

$$\mathsf{unit} : \mathsf{Id} \Rightarrow \mathsf{List}$$
$$\mathsf{unit}_X : \mathsf{Id}_0\, X \to \mathsf{List}_0\, X$$
$$\mathsf{unit}_X : X \to [X]$$
$$\mathsf{unit}_X\, x := [x]$$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\mathsf{unit}_X"] \arrow[d, "f"'] & {[X]} \arrow[d, "\mathsf{List}_{X,Y}\, f"] \\ Y \arrow[r, "\mathsf{unit}_Y"'] & {[Y]} \end{tikzcd}$$

For each $f : X \to Y$ in $\mathbf{Set}$, the above diagram commutes by naturality.

Examples of Natural Transformations

Example

We define the natural transformation flatten on the functors $\mathsf{List} \circ \mathsf{List}, \mathsf{List} : \mathbf{Set} \to \mathbf{Set}$.

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "\mathsf{List} \circ \mathsf{List}"] \arrow[r, bend right, "\mathsf{List}"'] \arrow[r, phantom, "\mathsf{flatten} \Downarrow"] & \mathbf{Set} \end{tikzcd}$$

(continued on next slide)

Examples of Natural Transformations

Example (continuation)

$$\mathsf{flatten} : \mathsf{List} \circ \mathsf{List} \Rightarrow \mathsf{List}$$

$$\mathsf{flatten}_X : \mathsf{List}_0\,(\mathsf{List}_0\,X) \to \mathsf{List}_0\,X$$

$$\mathsf{flatten}_X : [[X]] \to [X]$$

$$\mathsf{flatten}_X\,[[x^1_1, \ldots, x^1_{n_1}], \ldots, [x^k_1, \ldots, x^k_{n_k}]] := [x^1_1, \ldots, x^1_{n_1}, \ldots, x^k_1, \ldots, x^k_{n_k}]$$

Examples of Natural Transformations

Example (continuation)

$$\mathsf{flatten} : \mathsf{List} \circ \mathsf{List} \Rightarrow \mathsf{List}$$

$$\mathsf{flatten}_X : \mathsf{List}_0\,(\mathsf{List}_0\,X) \to \mathsf{List}_0\,X$$

$$\mathsf{flatten}_X : [[X]] \to [X]$$

$$\mathsf{flatten}_X\,[[x_1^1, \ldots, x_{n_1}^1], \ldots, [x_1^k, \ldots, x_{n_k}^k]] := [x_1^1, \ldots, x_{n_1}^1, \ldots, x_1^k, \ldots, x_{n_k}^k]$$

For each $f : X \to Y$ in $\mathbf{Set}$, the following diagram commutes by naturality.

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} {[[X]]} \arrow[r, "\mathsf{flatten}_X"] \arrow[d, "(\mathsf{List} \circ \mathsf{List})_{X,Y}\,f"'] & {[X]} \arrow[d, "\mathsf{List}_{X,Y}\,f"] \\ {[[Y]]} \arrow[r, "\mathsf{flatten}_Y"'] & {[Y]} \end{tikzcd}$$

Examples of Natural Transformations

Exercise 1

Verify naturality of reverse, unit and flatten.

Examples of Natural Transformations

Example

Let Id be the identity functor in $\mathbf{Set}$ and let $\times \circ (\mathsf{Id}, \mathsf{Id}) : \mathbf{Set} \to \mathbf{Set}$ the functor sending every set X to $X \times X$ and every function f to $f \times f$. We shall define the natural transformation **diagonal** Δ.

$$
\begin{tikzcd}
\mathbf{Set} \arrow[rr, bend left, "\mathsf{Id}"] \arrow[rr, bend right, "\times \circ (\mathsf{Id}, \mathsf{Id})"'] \arrow[rr, phantom, "\Delta \Downarrow"] & & \mathbf{Set}
\end{tikzcd}
$$

(continued on next slide)

Examples of Natural Transformations

Example (continuation)

$$\Delta : \mathsf{Id} \Rightarrow \times \circ (\mathsf{Id}, \mathsf{Id})$$

$$\Delta_X : \mathsf{Id}_0\, X \to (\times \circ (\mathsf{Id}, \mathsf{Id}))_0\, X$$

$$\Delta_X : X \to X \times X$$

$$\Delta_X\, x := (x, x)$$

Examples of Natural Transformations

Example (continuation)

$$\Delta : \mathsf{Id} \Rightarrow \times \circ (\mathsf{Id}, \mathsf{Id})$$
$$\Delta_X : \mathsf{Id}_0\, X \to (\times \circ (\mathsf{Id}, \mathsf{Id}))_0\, X$$
$$\Delta_X : X \to X \times X$$
$$\Delta_X\, x := (x, x)$$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[d, "f"'] & X \times X \arrow[d, "f \times f"] \\ Y \arrow[r, "\Delta_Y"'] & Y \times Y \end{tikzcd}$$

For each $f : X \to Y$ in $\mathbf{Set}$, the above diagram commutes by naturality.

Examples of Natural Transformations

Example

Let $\times, \pi_1 : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be the product and first projection functors, respectively. We shall define the natural transformation **first projection** π_1.

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, bend left, "\times"] \arrow[r, bend right, "\pi_1"'] \arrow[r, phantom, "\pi_1 \Downarrow"] & \mathcal{C} \end{tikzcd}$$

(continued on next slide)

Examples of Natural Transformations

Example (continuation)

$$\pi_1 : \times \Rightarrow \pi_1$$

$$\pi_{1(A,B)} : \times_0 (A, B) \to (\pi_1)_0 (A, B)$$

$$\pi_{1(A,B)} : A \times B \to A$$

Examples of Natural Transformations

Example (continuation)

$$\pi_1 : \times \Rightarrow \pi_1$$

$$\pi_{1(A,B)} : \times_0 (A, B) \to (\pi_1)_0 (A, B)$$

$$\pi_{1(A,B)} : A \times B \to A$$

$$\begin{tikzcd}
(A, B) \arrow[d, "{(f,g)}"'] & A \times B \arrow[r, "\pi_{1(A,B)}"] \arrow[d, "f \times g"'] & A \arrow[d, "f"] \\
(A', B') & A' \times B' \arrow[r, "\pi_{1(A',B')}"'] & A'
\end{tikzcd}$$

For each $(f, g) : (A, B) \to (A', B')$ in $\mathcal{C} \times \mathcal{C}$, the above diagram commutes by naturality.

Examples of Natural Transformations

Example

Let $\mathcal{C}$ be a category with terminal object 1, let Id be the identity functor in $\mathcal{C}$ and let $K_1 : \mathcal{C} \to \mathcal{C}$ be the functor mapping all objects of $\mathcal{C}$ to 1 and all arrows of $\mathcal{C}$ to id_1. We shall define the natural transformation κ.

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "\mathsf{Id}"] \arrow[rr, bend right, "K_1"'] \arrow[rr, phantom, "\kappa \Downarrow"] & & \mathcal{C}
\end{tikzcd}
$$

(continued on next slide)

Examples of Natural Transformations

Example (continuation)

$$\begin{aligned} \kappa &: \mathsf{Id} \Rightarrow K_1 \\ \kappa_A &: \mathsf{Id}_0\, A \to (K_1)_0\, A \\ \kappa_A &: A \to 1 \end{aligned}$$

Examples of Natural Transformations

Example (continuation)

$$\begin{aligned} \kappa &: \mathsf{Id} \Rightarrow K_1 \\ \kappa_A &: \mathsf{Id}_0\, A \to (K_1)_0\, A \\ \kappa_A &: A \to 1 \end{aligned}$$

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\kappa_A"] \arrow[d, "f"'] & 1 \arrow[d, "\mathrm{id}_1"] \\ B \arrow[r, "\kappa_B"'] & 1 \end{tikzcd}$$

For each $f : A \to B$ in $\mathcal{C}$, the above diagram commutes by naturality.

Exercises

Exercise 2

Verify naturality of the natural transformation κ.

Natural Isomorphisms

Natural Isomorphisms

Definition

A natural transformation

$$\begin{aligned} \tau &: F \Rightarrow G \\ \tau_A &: F_0\, A \to G_0\, A, \quad \text{for all } A \text{ in } \mathcal{C} \end{aligned}$$

is a **natural isomorphism** iff each τ_A is an isomorphism.

Natural Isomorphisms

Definition

A natural transformation

$$\tau : F \Rightarrow G$$
$$\tau_A : F_0\,A \to G_0\,A, \quad \text{for all } A \text{ in } \mathcal{C}$$

is a **natural isomorphism** iff each τ_A is an isomorphism.

Notation

A natural isomorphism is denoted by $\tau : F \overset{\cong}{\Longrightarrow} G$.

Natural Isomorphisms

Example

Let $\mathcal{C}$ be a category with products and let the functors (textbook and [Awodey 2010, § 7.4])

$$-_1 \times (-_2 \times -_3) : \mathcal{C}^3 \to \mathcal{C} \qquad \text{and} \qquad (-_1 \times -_2) \times -_3 : \mathcal{C}^3 \to \mathcal{C}.$$

The natural isomorphism a shows that the product is associative.

$$\begin{aligned} a &: -_1 \times (-_2 \times -_3) \overset{\cong}{\Longrightarrow} (-_1 \times -_2) \times -_3 \\ a_{A,B,C} &: A \times (B \times C) \xrightarrow{\cong} (A \times B) \times C \\ a_{A,B,C} &:= \langle \langle \pi_1, \pi_1 \circ \pi_2 \rangle, \pi_2 \circ \pi_2 \rangle. \end{aligned}$$

Example

Let $\mathcal{C}$ be a category with binary products. We define the functor $\bar{\times} : \mathcal{C}^2 \to \mathcal{C}$ by (textbook and [Awodey 2010, Example 7.8])

$$\bar{\times}_0\,(A, B) := \times_0\,(B, A) = B \times A, \qquad \bar{\times}_1\,(f, g) := \times_1\,(g, f) = g \times f.$$

The natural isomorphism s shows that the product is symmetric.

$$s : \times \overset{\cong}{\Longrightarrow} \bar{\times}$$
$$s_{A,B} : A \times B \xrightarrow{\cong} B \times A$$
$$s_{A,B} := \langle \pi_2, \pi_1 \rangle.$$

$$\begin{tikzcd}
(A, B) \arrow[d, "{(f, g)}"'] \\
(A', B')
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times B \arrow[r, "s_{A,B}"] \arrow[d, "f \times g"'] & B \times A \arrow[d, "g \times f"] \\
A' \times B' \arrow[r, "s_{A',B'}"'] & B' \times A'
\end{tikzcd}$$

Natural Isomorphisms

Example

Let $\mathcal{C}$ be a category with binary products and terminal object 1 and let the functors

$$\begin{aligned} &1 \times - : \mathcal{C} \to \mathcal{C} \\ &(1 \times -)_0\, A := 1 \times A \\ &(1 \times -)_1\, f := \langle \mathrm{id}_1, f \rangle, \end{aligned} \qquad \begin{aligned} &- \times 1 : \mathcal{C} \to \mathcal{C} \\ &(- \times 1)_0\, A := A \times 1 \\ &(- \times 1)_1\, f := \langle f, \mathrm{id}_1 \rangle. \end{aligned}$$

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Example (continuation)

The natural isomorphisms l and r show that 1 is the unit of the product.

$$\begin{aligned} l &: 1 \times - \overset{\cong}{\Longrightarrow} \mathsf{Id} \\ l_A &: 1 \times A \overset{\cong}{\longrightarrow} A \\ l_A &:= \pi_2, \end{aligned}$$

$$\begin{tikzcd} A \arrow[d, "f"'] & 1 \times A \arrow[r, "l_A"] \arrow[d, "\langle \mathrm{id}_1, f \rangle"'] & A \arrow[d, "f"] \\ B & 1 \times B \arrow[r, "l_B"'] & B \end{tikzcd}$$

$$\begin{aligned} r &: - \times 1 \overset{\cong}{\Longrightarrow} \mathsf{Id} \\ r_A &: A \times 1 \overset{\cong}{\longrightarrow} A \\ r_A &:= \pi_1. \end{aligned}$$

$$\begin{tikzcd} A \arrow[d, "f"'] & A \times 1 \arrow[r, "r_A"] \arrow[d, "\langle f, \mathrm{id}_1 \rangle"'] & A \arrow[d, "f"] \\ B & B \times 1 \arrow[r, "r_B"'] & B \end{tikzcd}$$

Natural Isomorphisms

Exercise 3

Verify that the families of arrows $s_{A,B}$, l_A and r_A, from the previous examples, are natural isomorphisms (textbook, Exercise 53).

Natural Isomorphisms

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Verify that the families of arrows $s_{A,B}$, l_A and r_A, from the previous examples, are natural isomorphisms (textbook, Exercise 53).

Remark

Because natural isomorphisms are a self-dual notion, we get also natural isomorphisms s, l and r for a category with binary coproducts and initial object.

Natural Transformations Between Hom-Functors

Natural Transformations Between Hom-Functors

Definition

Let $\mathcal{C}$ be a (locally small) category and let A and B be objects of $\mathcal{C}$. Recall the hom-functors

$$\mathcal{C}(A,-) : \mathcal{C} \to \mathbf{Set} \qquad \text{(covariant)}$$
$$\mathcal{C}(-,B) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set} \qquad \text{(contravariant)}$$

(continued on next slide)

Natural Transformations Between Hom-Functors

Definition (continuation, first notation)

Let $f : A \to B$ in $\mathcal{C}$. We define the natural transformation $\mathcal{C}(f,-)$ and we show its naturality condition for $h : C \to D$ in $\mathcal{C}$.

$$\mathcal{C}(f,-) : \mathcal{C}(B,-) \Rightarrow \mathcal{C}(A,-)$$
$$\mathcal{C}(f,-)_C : \mathcal{C}(B,C) \to \mathcal{C}(A,C)$$
$$\mathcal{C}(f,-)_C\, g : \mathcal{C}(A,C)$$
$$\mathcal{C}(f,-)_C\, g := g \circ f$$

$$\begin{tikzcd} C \arrow[d, "h"'] \\ D \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(B,C) \arrow[r, "\mathcal{C}(f,-)_C"] \arrow[d, "\mathcal{C}(B,-)_1\, h"'] & \mathcal{C}(A,C) \arrow[d, "\mathcal{C}(A,-)_1\, h"] \\ \mathcal{C}(B,D) \arrow[r, "\mathcal{C}(f,-)_D"'] & \mathcal{C}(A,D) \end{tikzcd}$$

where $\mathcal{C}(f,-)_C = \mathcal{C}(-,C)_0\, f$.

Natural Transformations Between Hom-Functors

Definition (continuation, second notation)

Let $f : A \to B$ in $\mathcal{C}$. We define the natural transformation $\mathcal{C}(f,-)$ and we show its naturality condition for $h : C \to D$ in $\mathcal{C}$.

$$\mathcal{C}(f,-) : \mathcal{C}(B,-) \Rightarrow \mathcal{C}(A,-)$$
$$\mathcal{C}(f,-)_C : \mathcal{C}(B,C) \to \mathcal{C}(A,C)$$
$$\mathcal{C}(f,-)_C\, g : \mathcal{C}(A,C)$$
$$\mathcal{C}(f,-)_C\, g := g \circ f$$

$$\begin{tikzcd} C \arrow[d, "h"'] \\ D \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(B,C) \arrow[r, "\mathcal{C}(f,-)_C = \mathcal{C}(f,C)"] \arrow[d, "\mathcal{C}(B,h)"'] & \mathcal{C}(A,C) \arrow[d, "\mathcal{C}(A,h)"] \\ \mathcal{C}(B,D) \arrow[r, "\mathcal{C}(f,-)_C = \mathcal{C}(f,D)"'] & \mathcal{C}(A,D) \end{tikzcd}$$

Natural Transformations Between Hom-Functors

Exercise 4

Define the natural transformation $\mathcal{C}(-, f) : \mathcal{C}(-, A) \Rightarrow \mathcal{C}(-, B)$ and verify its naturality (textbook, Exercise 55).

Yoneda Lemma

Yoneda embedding

Let $\mathcal{C}$ be a locally small category and let A and B be objects of $\mathcal{C}$. For each natural transformation between hom-functors $\tau : \mathcal{C}(A, -) \to \mathcal{C}(B, -)$, there is a unique arrow $f : B \to A$ such that

$$\tau = \mathcal{C}(f, -).$$

Yoneda Lemma

Yoneda embedding

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$$\tau = \mathcal{C}(f, -).$$

Remark

The Yoneda lemma is a generalisation of the Yoneda embedding.

Compositions of Natural Transformations

Vertical Composition

Definition

Let $\mathcal{C}$ be a category, let $F, G, H : \mathcal{C} \to \mathcal{D}$ be functors, and let $\tau : F \Rightarrow G$ and $\mu : G \Rightarrow H$ be natural transformations. The **vertical composition** of μ and τ is the natural transformation $\mu \circ \tau$ defined by

$$\mu \circ \tau : F \Rightarrow H$$
$$(\mu \circ \tau)_A : F_0\, A \to H_0\, A := \mu_A \circ \tau_A, \quad \text{for all } A \text{ in } \mathcal{C}.$$

(continued on next slide)

Vertical Composition

Definition (continuation)

That is, for all $f : A \to B$ in $\mathcal{C}$ the following diagram commutes.

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} F_0\, A \arrow[r, "F_1\, f"] \arrow[d, "\tau_A"'] & F_0\, B \arrow[d, "\tau_B"] \\ G_0\, A \arrow[r, "G_1\, f"'] \arrow[d, "\mu_A"'] & G_0\, B \arrow[d, "\mu_B"] \\ H_0\, A \arrow[r, "H_1\, f"'] & H_0\, B \end{tikzcd}$$

$$\begin{pmatrix} (G_1\, f) \circ \tau_A = \tau_B \circ (F_1\, f) \\ (H_1\, f) \circ \mu_A = \mu_B \circ (G_1\, f) \\ (H_1\, f) \circ (\mu \circ \tau)_A = (\mu \circ \tau)_B \circ (F_1\, f) \end{pmatrix}$$

Vertical Composition

Exercise 5

Let $\mathcal{C}$ be a category, let $F, G, H : \mathcal{C} \to \mathcal{D}$ be functors, and let $\tau : F \Rightarrow G$ and $\mu : G \Rightarrow H$ be natural transformations. Show that $\mu \circ \tau : F \Rightarrow H$ is a natural transformation.

Functor Category

Functor Category

Introduction

Since we can define an associative composition of natural transformations and this composition has an identity natural transformation, we can define a category where the objects are functors and the arrows are natural transformations.

Functor Category

Definition

Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be an arbitrary category. The **functor category** $\mathrm{Func}(\mathcal{C}, \mathcal{D})$ is defined by

(i) Objects: Functors $F : \mathcal{C} \to \mathcal{D}$.

(ii) Arrows: Natural transformations $\tau : F \Rightarrow G$.

Functor Category

Definition

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(ii) Arrows: Natural transformations $\tau : F \Rightarrow G$.

(iii) Composition of arrows: Vertical composition of natural transformations.

Functor Category

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(i) Objects: Functors $F : \mathcal{C} \to \mathcal{D}$.

(ii) Arrows: Natural transformations $\tau : F \Rightarrow G$.

(iii) Composition of arrows: Vertical composition of natural transformations.

(iv) Identity arrow

$$\begin{aligned} \mathrm{id}_F &: F \to F \\ (\mathrm{id}_F)_A &: F_0\, A \to F_0\, A \quad \text{for all } A \text{ in } \mathcal{C}, \\ (\mathrm{id}_F)_A &:= \mathrm{id}_{(F_0\, A)}. \end{aligned}$$

References

References

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Awodey, S. [2006] (2010). Category Theory. 2nd ed. Vol. 52. Oxford Logic Guides. Oxford University Press (cit. on pp. 34, 35).

Mac Lane, S. [1971] (1998). Categories for the Working Mathematician. 2nd ed. Springer (cit. on pp. 5, 6).