Category Theory and Functional Programming
Introduction

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, pages, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Abramsky and Tzevelekos 2011].

Outline

From Set Theory to Category Theory

‘Algebra’ of Functions

Definition

Let $f : X \to Y$ and $g : Y \to Z$ be two functions. The **composite of g after f** is the function defined by

$$g \circ f : X \to Z := x \mapsto g\,(f\,x\,).$$

‘Algebra’ of Functions

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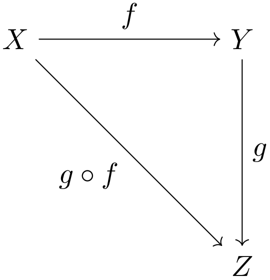
Diagram.



‘Algebra’ of Functions

Remark

The textbook writes ‘$g \circ f(x)$’ instead of ‘$(g \circ f)\, x$’.

'Algebra' of Functions

Theorem

Let $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$ be three functions. Then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

That is, the composition of functions is associative.

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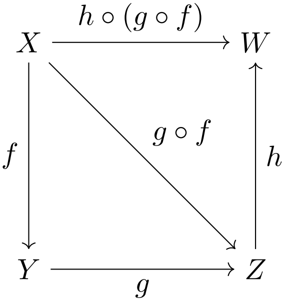
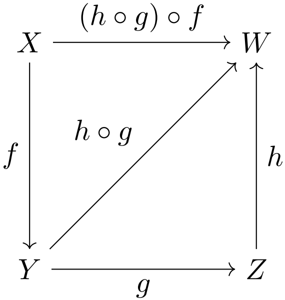
‘Algebra’ of Functions

Theorem (continuation)

Diagrams.

(i)





$$h \circ (g \circ f) = (h \circ g) \circ f$$

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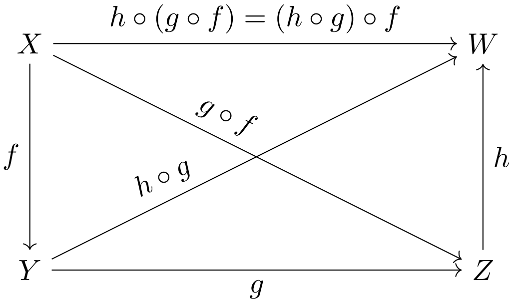
‘Algebra’ of Functions

Theorem (continuation)

Diagrams.

(ii) In [Mac Lane 1998, p. 8].



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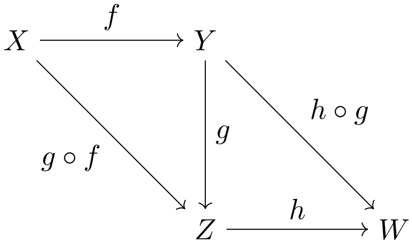
‘Algebra’ of Functions

Theorem (continuation)

Diagrams.

(iii) In [Awodey 2010, p. 3].



$$h \circ (g \circ f) = (h \circ g) \circ f$$

‘Algebra’ of Functions

Definition

Let X be a set. The **identity function on X** is defined by

$$\mathrm{id}_X : X \to X := x \mapsto x.$$

‘Algebra’ of Functions

Theorem

Let $f : X \to Y$ be a function. Then

$$f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f.$$

That is, the identity functions are the unit for composition.

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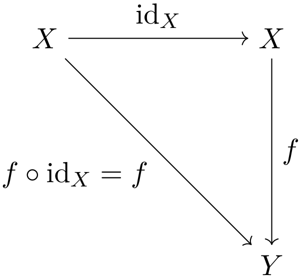
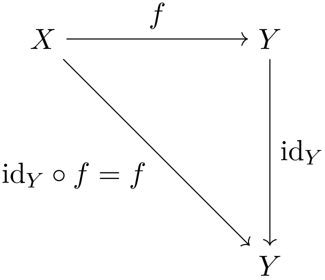
‘Algebra’ of Functions

Theorem (continuation)

Diagrams.

(i)





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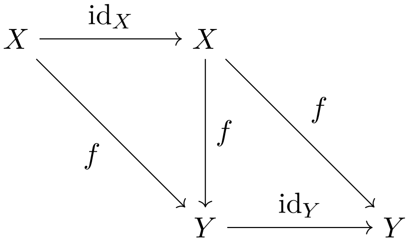
Theorem (continuation)

Diagrams.

(ii) In [Awodey 2010, p. 4].



$$f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$$

From Elements to Functions

Elements as functions

Let $\mathbb{1} := \{*\}$ be an one-element set and let X be a set. For each $x \in X$ we define the function

$$\overline{x} : \mathbb{1} \to X := * \mapsto x.$$

From Elements to Functions

Elements as functions

Let $\mathbb{1} := \{*\}$ be an one-element set and let X be a set. For each $x \in X$ we define the function

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Theorem

Let X be a set. The set X and the set of functions $\{\, \overline{x} : \mathbb{1} \to X \mid x \in X \,\}$ are isomorphic.

From Set Theory to Category Theory

Definition

Let $f : X \to Y$ be a function. The function f is

injective	iff	for all $x, x' \in X$, $f\,x = f\,x'$ implies $x = x'$
surjective	iff	for all $y \in Y$, there exists $x \in X$ such that $f\,x = y$
monic	iff	for all $g, h : Z \to X$, $f \circ g = f \circ h$ implies $g = h$
epic	iff	for all $i, j : Y \to Z$, $i \circ f = j \circ f$ implies $i = j$

From Set Theory to Category Theory

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epic	iff	for all $i, j : Y \to Z$, $i \circ f = j \circ f$ implies $i = j$

Remark

Nouns: Injection, surjection, monomorphism and epimorphism.

From Set Theory to Category Theory

Theorem (Proposition 1)

Let $f : X \to Y$. Then,

(i) the function f is injective iff f is monic,

(ii) the function f is surjective iff f is epic.

From Set Theory to Category Theory

Theorem (Proposition 1)

Let $f : X \to Y$. Then,

(i) the function f is injective iff f is monic,

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Exercise 1

Let $f : X \to Y$ be a function. Show that f is injective iff it is monic (Proposition 1.i).

Exercise 2

Let $f : X \to Y$ be a function. Show that f is surjective iff it is epic (Exercise 2).

From Functional Programming to Category Theory

From Functional Programming to Category Theory

Types, composition, identities, applicative and functional laws

Whiteboard.

Applicative laws

$$
\begin{aligned}
\mathrm{id}\, x &= x, \\
(g \circ f)\, x &= g\, (f\, x), \\
\mathrm{fst}\, (x, y) &= x, \\
\langle f, g \rangle\, x &= (f\, x, g\, x).
\end{aligned}
$$

Definition of a Category

Definition of a Category

Definition

A **category** $\mathcal{C}$ consists of:

(i) A collection $\mathrm{Obj}(\mathcal{C})$ of **objects**.

Notation. Objects are denoted by $A, B, C, \ldots$

(ii) A collection $\mathrm{Ar}(\mathcal{C})$ of **arrows** or **morphisms**.

Notation. Arrows are denoted by $f, g, h, \ldots$

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Definition of a Category

Definition (continuation)

(iii) Two mappings

$$\begin{aligned} \mathrm{dom} : \mathrm{Ar}(\mathcal{C}) &\to \mathrm{Obj}(\mathcal{C}) && \text{(source)}, \\ \mathrm{cod} : \mathrm{Ar}(\mathcal{C}) &\to \mathrm{Obj}(\mathcal{C}) && \text{(target)}. \end{aligned}$$

Definition of a Category

Definition (continuation)

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These mappings assign to each arrow f its **domain** $\operatorname{dom} f$ and its **codomain** $\operatorname{cod} f$.

Definition of a Category

Definition (continuation)

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These mappings assign to each arrow f its **domain** $\mathrm{dom}\, f$ and its **codomain** $\mathrm{cod}\, f$.

Notation. An arrow f with $\mathrm{dom}\, f = A$ and $\mathrm{cod}\, f = B$ is written $A \xrightarrow{f} B$ or $f : A \to B$.

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Definition of a Category

Definition (continuation)

Notation. The collection $\mathcal{C}(A, B)$ is the collection of arrows from object A to object B, that is,

$$\mathcal{C}(A, B) := \left\{ f \in \mathrm{Ar}(\mathcal{C}) \;\middle|\; A \xrightarrow{f} B \right\}.$$

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Definition of a Category

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Convention. All the collections $\mathcal{C}(A, B)$ are hom-sets in the textbook.

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Definition of a Category

Definition (continuation)

(iv) For all objects A,B,C, a **composition** map

$$\mathcal{C}_{A,B,C} : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C).$$

Notation. The map $\mathcal{C}_{A,B,C}\,(f, g)$ is written $g \circ f$.

Definition of a Category

Definition (continuation)

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(v) For all object A, an **identity** arrow

$$A \xrightarrow{\mathrm{id}_A} A.$$

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Definition of a Category

Definition (continuation)

The above items must satisfy the following axioms, where arrow equality is a logical primitive.

Definition of a Category

Definition (continuation)

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(i) Associativity law

For all arrows $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $C \xrightarrow{h} D$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Definition (continuation)

The above items must satisfy the following axioms, where arrow equality is a logical primitive.

(i) Associativity law

For all arrows $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $C \xrightarrow{h} D$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(ii) Unit laws

For all arrow $A \xrightarrow{f} B$,

$$f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f.$$

Definition of a Category

Remark

Some authors[†] state the unit laws in the following equivalent way:

For all arrows $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$,

$$\begin{aligned} \mathrm{id}_B \circ f &= f, \\ g \circ \mathrm{id}_B &= g. \end{aligned}$$

[†] E.g. [Asperti and Longo 1980; Goldblatt 2006; Mac Lane 1998].

Definition of a Category

Remark

Note that the axioms in the definition of category are generalised monoid axioms.

Diagrams in Categories

Diagrams in Categories

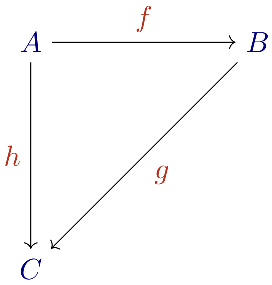
Commutativity of diagrams

A diagram commutes when every possible path from one object to other object is the same.

Diagrams in Categories

Basic cases

(i) Commutativity of a triangle

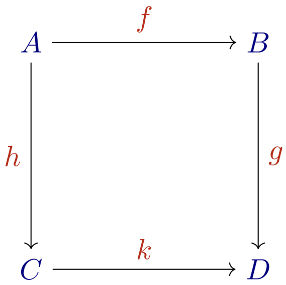


$$\left(h = g \circ f\right)$$

Diagrams in Categories

Basic cases

(v) Commutativity of a square

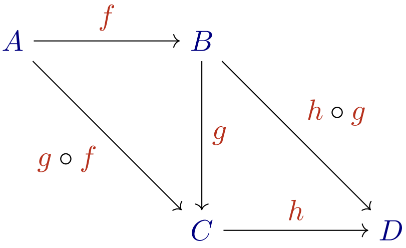


$$\left(g \circ f = k \circ h\right)$$

Diagrams in Categories

Example

Let $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ and $C \xrightarrow{h} D$. The associativity of the composition is equivalent to say that the following diagram commutes.

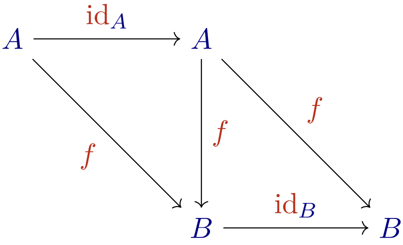


$$\big(h \circ (g \circ f) = (h \circ g) \circ f\big)$$

Diagrams in Categories

Example

Let $A \xrightarrow{f} B$. The unit of the identity arrow is equivalent to say that the following diagram commutes.



$$\left(f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f\right)$$

Examples of Categories

Example

The category $\mathbf{Set}$ of sets and functions.

Examples of Categories

Example

Mathematical structures and structure preserving functions.

- **Pos** (partially ordered sets and monotone functions)
- **Mon** (monoids and monoid homomorphisms)
- **Grp** (groups and group homomorphisms)
- **Top** (topological spaces and continuous functions)

Examples of Categories

Example

Mathematical structures and structure preserving functions.

- $\mathbf{Pos}$ (partially ordered sets and monotone functions)
- $\mathbf{Mon}$ (monoids and monoid homomorphisms)
- $\mathbf{Grp}$ (groups and group homomorphisms)
- $\mathbf{Top}$ (topological spaces and continuous functions)

Exercise 3

Show that $\mathbf{Pos}$, $\mathbf{Mon}$, $\mathbf{Grp}$ and $\mathbf{Top}$ are categories (Exercise 6).

Examples of Categories

Remark

The arrows of a category do no have to be functions as shows the following example.

Examples of Categories

Example

The category $\mathbf{Rel}$.

- The objects are sets.
- The arrows $X \xrightarrow{R} Y$ are the relations $R \subseteq X \times Y$.
- The arrow composition is the relation composition. Given $X \xrightarrow{R} Y$ and $Y \xrightarrow{S} Z$ then

$$S \circ R := \{\, (x, z) \in X \times Z \mid \text{there exists } y \in Y \text{ such as } (x, y) \in R \text{ and } (y, z) \in S \,\}.$$

- The identity arrow on X is the equality relation on X, that is

$$\mathrm{id}_X := \{\, (x, x) \in X \times X \mid x \in X \,\}.$$

Examples of Categories

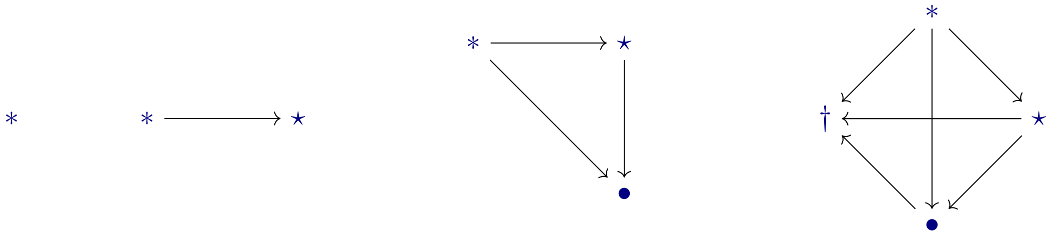
Remark

The objects of a category do no have to be sets as show the following examples.

Examples of Categories

Example

The categories $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{4}$. The diagrams do not show the identity arrows.



$\mathbf{1}$ $\mathbf{2}$ $\mathbf{3}$ $\mathbf{4}$

Examples of Categories

Example

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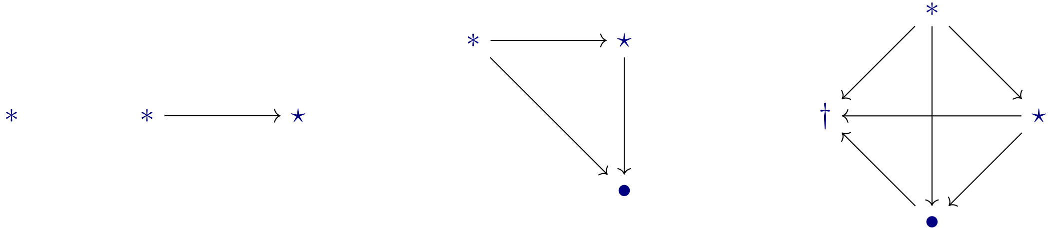


$\mathbf{1}$ $\mathbf{2}$ $\mathbf{3}$ $\mathbf{4}$

Remark

The category $\mathbf{n}$ has $n(n+1)/2$ arrows [Zeng n.d.].

Examples of Categories

Example

The empty category. It has no objects nor arrows.

Examples of Categories

Example

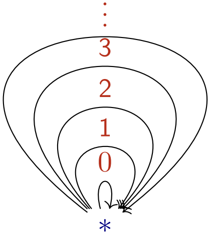
Any monoid is a one-object category.

- Arrows: Elements of the monoid
- Composition: Monoid binary operation
- Identity arrow: Monoid unit

Examples of Categories

Example

One-object category from monoid $(\mathbb{N}, +, 0)$.



$$\begin{pmatrix} 0 + n = n \\ 1 + 1 = 2 \\ 1 + 2 = 3 \\ \vdots \end{pmatrix}$$

Examples of Categories

Example

Any pre-ordered set $(P, \preceq)$ is a category.

- Objects: Elements of P
- Arrows: There is an arrow $A \to B$ iff $A \preceq B$
- Composition: Binary relation $\preceq$
- Identity arrow: The arrow $A \to A$ because $A \preceq A$

Examples of Categories

Example

Any pre-ordered set $(P, \preceq)$ is a category.

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Remark

Note that the above category has at most one arrow between any two objects.

Examples of Categories

Example

Any category with at most one arrow between any two objects is a pre-order.

- Elements of the pre-order: Objects of the category
- Binary relation: $A \preceq B$ iff there is an arrow $A \to B$

  The relation $\preceq$ is transitive because the composition of functions and it is reflexive because the identity arrows.

Examples of Categories

Example

A category for a simple functional programming language given by (adapted from [Pierce 1991]):

- Types: $\mathtt{Nat}$, $\mathtt{Bool}$, $\mathtt{Unit}$, $\cdot \to \cdot$
- Built-in functions:

$$\begin{aligned} \mathtt{isZero} &: \mathtt{Nat} \to \mathtt{Bool} && \text{(test for zero)} \\ \mathtt{not} &: \mathtt{Bool} \to \mathtt{Bool} && \text{(negation)} \\ \mathtt{succ} &: \mathtt{Nat} \to \mathtt{Nat} && \text{(successor)} \end{aligned}$$

- Constants

$$\mathtt{zero} : \mathtt{Nat}; \quad \mathtt{true}, \mathtt{false} : \mathtt{Bool}; \quad \mathtt{unit} : \mathtt{Unit}.$$

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Examples of Categories

Example (continuation)

The category is given by:

- Objects: Types
- Arrows:
  - Built-in functions
  - The constants are arrows from `Unit` to the type of the constant
  - Add arrows required by arrow composition
- Identity arrows: Identity functions in each type
- Equating arrows that represent the same functions (according to the semantics of the language)

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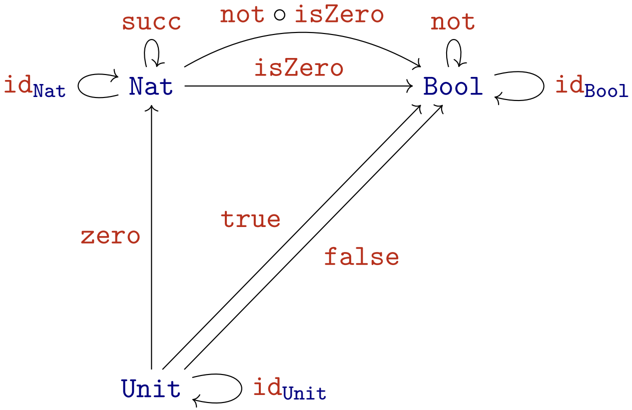
Examples of Categories

Example (continuation)



Same functions

$$\begin{pmatrix} \texttt{not} \circ \texttt{true} = \texttt{false} \\ \texttt{not} \circ \texttt{false} = \texttt{true} \\ \texttt{isZero} \circ \texttt{zero} = \texttt{true} \\ \texttt{isZero} \circ \texttt{succ} = \texttt{false} \\ \texttt{unit} = \texttt{id}_{\texttt{Unit}} \end{pmatrix}$$

Examples of Categories

Exercise 4

Show an example of a category from logic. See, e.g. [Awodey 2010, § 1.14. Example 10].

Examples of Categories

Example

Hask is the *idealised* category for the Haskell programming language.

- Objects: Haskell's (unlifted) types
- Arrows: Haskell's functions
- Composition:

```
(.) :: (b -> c) -> (a -> b) -> a -> c
g . f = \x -> g (f x)
```

- Identity arrow:

```
id :: a -> a
id x = x
```

Examples of Categories

Exercise 5

Given some implementation of categories in Haskell, show two examples of categories in that implementation.

Isomorphisms

Monomorphisms

Definition

Let $\mathcal{C}$ be a category and let $A \xrightarrow{f} B$ be an arrow in $\mathcal{C}$. The arrow f is **monic** (or a **monomorphism**) iff

$$\text{for all } C \xrightarrow{g,h} A,\ f \circ g = f \circ h \text{ implies } g = h,$$

that is,

$$C \underset{h}{\overset{g}{\rightrightarrows}} A \xrightarrow{f} B \qquad \text{implies} \qquad g = h,$$

where the above diagram commutes.

Epimorphisms

Definition

Let $\mathcal{C}$ be a category and let $A \xrightarrow{f} B$ be an arrow in $\mathcal{C}$. The arrow f is **epic** (or a **epimorphism**) iff

$$\text{for all } B \xrightarrow{i,j} C,\ i \circ f = j \circ f \text{ implies } i = j,$$

that is,

$$A \xrightarrow{f} B \underset{j}{\overset{i}{\rightrightarrows}} C \qquad \text{implies} \qquad i = j,$$

where the above diagram commutes.

Isomorphisms

Definition

Let $\mathcal{C}$ be a category. An arrow $A \xrightarrow{i} B$ in $\mathcal{C}$ is an **isomorphism** (or **iso**) iff there exists an arrow $B \xrightarrow{j} A$ in $\mathcal{C}$ such that

$$j \circ i = \mathrm{id}_A \quad \text{and} \quad i \circ j = \mathrm{id}_B .$$

Isomorphisms

Definition

Let $\mathcal{C}$ be a category. An arrow $A \xrightarrow{i} B$ in $\mathcal{C}$ is an **isomorphism** (or **iso**) iff there exists an arrow $B \xrightarrow{j} A$ in $\mathcal{C}$ such that

$$j \circ i = \mathrm{id}_A \quad \text{and} \quad i \circ j = \mathrm{id}_B.$$

The arrow j is the **inverse** of i and it is denoted by i^{-1}.

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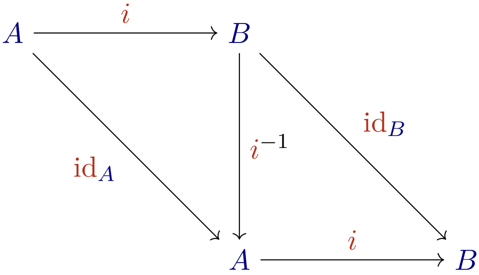
Isomorphisms

Definition (continuation)

That is, an arrow $A \xrightarrow{i} B$ is an isomorphism iff there exists an arrow $B \xrightarrow{i^{-1}} A$ such that the following diagram commutes



$$\begin{pmatrix} i^{-1} \circ i = \mathrm{id}_A \\ i \circ i^{-1} = \mathrm{id}_B \end{pmatrix}$$

Isomorphisms

Notation

An isomorphism $i : A \to B$ is denoted by $i : A \xrightarrow{\cong} B$.

Isomorphisms

Notation

An isomorphism $i : A \to B$ is denoted by $i : A \xrightarrow{\cong} B$.

Definition

Two objects A and B are **isomorphic**, written $A \cong B$, iff there exists $i : A \xrightarrow{\cong} B$.

Isomorphisms

Theorem

If an arrow has inverse it is unique.

Exercise 6

Proof the previous theorem (Exercise 10).

Isomorphisms

Exercise 7

Show that $\cong$ is an equivalence relation on the objects of a category (Exercise 11).

Isomorphisms

Example

Isomorphisms in $\mathbf{Set}$ and $\mathbf{Rel}$ correspond to one-one correspondences (bijections).

Isomorphisms

Example

Isomorphisms in **Grp** correspond to group isomorphisms, in **Pos** to order isomorphisms and in **Top** to homeomorphisms.

Isomorphisms

Example

Recall that any monoid is a one-object category. Any group is a one-object category in which every arrow is an isomorphism.

Isomorphisms

Example

Recall that any monoid is a one-object category. Any group is a one-object category in which every arrow is an isomorphism.

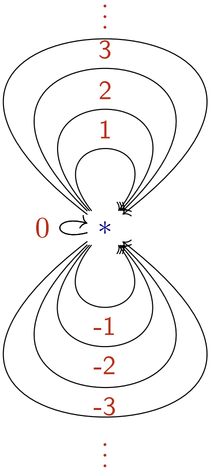
Exercise 8

Verify the previous example.

Isomorphisms

Example

One-object category from monoid $(\mathbb{Z}, +, 0)$.



$$\begin{pmatrix} 0 + n = n \\ 1 + 1 = 2 \\ 1 + 2 = 3 \\ \vdots \\ 1 + -1 = 0 \\ 2 + -2 = 0 \\ \vdots \end{pmatrix}$$

Groupoids

Definition

A **groupoid** is a category in which every arrow is an isomorphism.

Groupoids

Example

A group is one-object grupoid.

Groupoids

Definition

A **setoid** $(X, \sim)$ is a set X equipped with an equivalence relation $\sim$.

Groupoids

Definition

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Example

Given a setoid $(X, \sim)$ we can define an associated grupoid.

- Objects: Elements of X
- Arrows: There is an arrow $x \to y$ iff $x \sim y$.
- Composition: From transitivity of $\sim$.
- Identity arrow: From reflexivity of $\sim$.

Monics, Epics and Isos

Theorem (Awodey [2010, Proposition 2.9])

If an arrow is iso then it is monic and epic.

Monics, Epics and Isos

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Exercise 9

Proof the previous theorem.

Monics, Epics and Isos

Example (Exercise 1.1.6.e)

In the category $\mathbf{Mon}$ of monoids and monoid homomorphisms, consider the inclusion map

$$i : (\mathbb{N}, +, 0) \to (\mathbb{Z}, +, 0)$$

of natural numbers into the integers. Show that this arrow is both monic and epic. Is it an iso?

Monics, Epics and Isos

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Solution

Whiteboard.

Monics, Epics and Isos

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Solution

Whiteboard.

Remark

As showed the previous exercises if an arrow is monic and epic does not imply that it is an iso.

Skeletal Categories

Definition (Awodey [2010])

A category is **skeletal** iff isomorphic objects are always equals.

Opposite Categories and Duality

Opposite Categories and Duality

Introduction

We get a category from other category by turning around the arrows and then we get a duality principle between both categories.

Opposite Categories

Definition

Let $\mathcal{C}$ be a category. The **opposite** (or **dual**) category $\mathcal{C}^{\mathsf{op}}$ of $\mathcal{C}$ is defined by

$$\begin{aligned} \mathrm{Obj}(\mathcal{C}^{\mathsf{op}}) &:= \mathrm{Obj}(\mathcal{C}), \\ \mathcal{C}^{\mathsf{op}}(A^*, B^*) &:= \mathcal{C}(B, A), \\ \mathrm{id}_{A^*} &:= (\mathrm{id}_A)^*, \\ g^* \circ f^* &:= (f \circ g)^*, \end{aligned}$$

where we use * for distinguishing objects and arrows of the opposite category following [Awodey 2010].

Opposite Categories

Example

The left diagram in a category $\mathcal{C}$ corresponds to the right diagram in the category $\mathcal{C}^{\mathrm{op}}$.

The Duality Principle

Definition

Let S be a sentence. The dual statement S^{op} of S is the sentence obtained by reversing all the arrows of S.

Description

Let $\mathcal{C}$ be a category and S be a sentence. The **duality principle** states that

$$S \text{ holds in } \mathcal{C} \quad \text{iff} \quad S^{\text{op}} \text{ holds in } \mathcal{C}^{\text{op}}.$$

The Duality Principle

Example

Monic and epic are dual notions. That is, an arrow f is monic in $\mathcal{C}$ iff f^* is epic in $\mathcal{C}^{\mathsf{op}}$.

Subcategories

Subcategories

Definition

A **subcategory** $\mathcal{D}$ of a category $\mathcal{C}$ is a collection of some of the objects and arrows of $\mathcal{C}$

$$\mathrm{Obj}(\mathcal{D}) \subseteq \mathrm{Obj}(\mathcal{C}),$$
$$\mathrm{Ar}(\mathcal{D}) \subseteq \mathrm{Ar}(\mathcal{C}),$$

which is closed under dom, cod, id, and $\circ$, that is,

$$\begin{aligned}
f \in \mathrm{Ar}(\mathcal{D}) \quad &\text{implies} \quad \mathrm{dom}\, f, \mathrm{cod}\, f \in \mathrm{Obj}(\mathcal{D}), \\
f \in \mathcal{D}(A, B), g \in \mathcal{D}(B, C) \quad &\text{implies} \quad g \circ f \in \mathcal{D}(A, C), \\
A \in \mathrm{Obj}(\mathcal{D}) \quad &\text{implies} \quad \mathrm{id}_A \in \mathcal{D}(A, A).
\end{aligned}$$

(continued on next slide)

Subcategories

Definition (continuation)

Additionally, the category $\mathcal{D}$ is

- a **full subcategory** of $\mathcal{C}$ iff

$$\mathcal{D}(A, B) = \mathcal{C}(A, B), \qquad \text{for all } A, B \in \mathrm{Obj}(\mathcal{D}),$$

- a **lluf subcategory** of $\mathcal{C}$ iff

$$\mathrm{Obj}(\mathcal{D}) = \mathrm{Obj}(\mathcal{C}).$$

Subcategories

Example

$\mathbf{Grp}$ is a full subcategory of $\mathbf{Mon}$.

Subcategories

Example

$\mathbf{Grp}$ is a full subcategory of $\mathbf{Mon}$.

Example

$\mathbf{Set}$ is a lluf subcategory of $\mathbf{Rel}$.

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