Category Theory and Functional Programming
Functors

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, pages, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Abramsky and Tzevelekos 2011].

Outline

Introduction

Introduction

Question

What about of morphisms between categories?

Introduction

Question

What about of morphisms between categories?

Answer: Of course, them are functors.

Definition of a Functor

Definition of a Functor

Definition

A **(covariant) functor** $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is a mapping of objects to objects and arrows to arrows, that is,†

$$F_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}) \qquad \text{(object-map)}$$

$$F_1 : \mathrm{Ar}(\mathcal{C}) \to \mathrm{Ar}(\mathcal{D}) \qquad \text{(arrow-map)}$$

which for all objects A and arrows f and g, satisfies the **functoriality** conditions

$$F_1\,(g \circ f) = (F_1\,g) \circ (F_1\,f) \qquad \text{(preservation of compositions)}$$

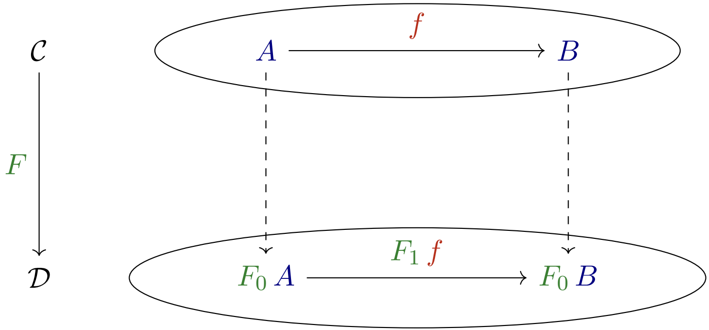
$$F_1\,\mathrm{id}_A = \mathrm{id}_{(F_0\,A)} \qquad \text{(preservation of identities)}$$

† The textbook does not use F_0 and F_1 but F.

Definition of a Functor

Remark

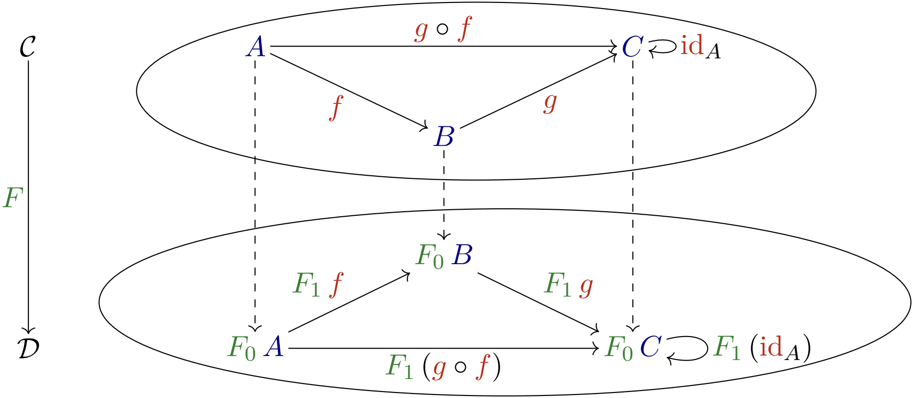
The functor $F : \mathcal{C} \to \mathcal{D}$ maps objects and arrows of $\mathcal{C}$ to objects and arrows of $\mathcal{D}$, respectively.



Definition of a Functor

Remark

The functor $F : \mathcal{C} \to \mathcal{D}$ preserves domains and codomains, identity arrows, and composition. It also maps each commutative diagram in $\mathcal{C}$ into a commutative diagram in $\mathcal{D}$.



Definition of a Functor

Remark

Given a functor $F : \mathcal{C} \to \mathcal{D}$, that is,

$$F_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}),$$
$$F_1 : \mathrm{Ar}(\mathcal{C}) \to \mathrm{Ar}(\mathcal{D}),$$

for all A, B in $\mathrm{Obj}(\mathcal{C})$, there is the map

$$F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(F_0\, A, F_0\, B),$$

and for all $f : A \to B$,

$$F_{A,B}\, f : F_0\, A \to F_0\, B.$$

Examples of Functors

Examples of Functors

Example

Let $\mathcal{P}\,S$ be the power set of the set S. The **(covariant) power set functor**

$$P : \mathbf{Set} \to \mathbf{Set}, \qquad \text{is defined by}$$

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$$P_0\,X := \mathcal{P}\,X$$

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$$P_0\,X := \mathcal{P}\,X$$

$$P_1 : \mathrm{Ar}(\mathbf{Set}) \to \mathrm{Ar}(\mathbf{Set})$$

$$P_{X,Y} : \mathbf{Set}(X, Y) \to \mathbf{Set}(P_0\,X, P_0\,Y)$$

$$P_{X,Y}\,f : \mathcal{P}\,X \to \mathcal{P}\,Y$$

$$P_{X,Y}\,f\,S := f(S) = \{\, f(x) \mid x \in S \,\}$$

(continued on next slide)

Examples of Functors

Example (continuation)

Let $X = \{0, 1\}$, $Y = \{\emptyset, X\}$ and $f : X \to Y$ defined by $f(0) = \emptyset$ and $f(1) = X$. Then,

$$P_0 : \mathrm{Obj}(\mathbf{Set}) \to \mathrm{Obj}(\mathbf{Set})$$
$$P_0\, X := \mathcal{P}\, X = \{\emptyset, \{0\}, \{1\}, X\},$$

$$P_{X,Y}\, f : \mathcal{P}\, X \to \mathcal{P}\, Y$$
$$P_{X,Y}\, f\, \emptyset := f(\emptyset) = \emptyset,$$
$$P_{X,Y}\, f\, \{0\} := f(\{0\}) = \{\emptyset\},$$
$$P_{X,Y}\, f\, \{1\} := f(\{1\}) = \{X\},$$
$$P_{X,Y}\, f\, \{0, 1\} := f(\{0, 1\}) = \{\emptyset, X\}.$$

Examples of Functors

Example

Let $(P, \preceq)$ and $(Q, \preceq)$ be two pre-orders seen as categories, denoted $\mathcal{P}$ and $\mathcal{Q}$, respectively. A functor $F : \mathcal{P} \to \mathcal{Q}$ is defined by

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$$F_0 : \mathrm{Obj}(\mathcal{P}) \to \mathrm{Obj}(\mathcal{Q})$$

$$F_1 : \mathrm{Ar}(\mathcal{P}) \to \mathrm{Ar}(\mathcal{Q})$$
$$F_{A,B} : \mathcal{P}(A, B) \to \mathcal{Q}(F_0\, A, F_0\, B)$$
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Since $\mathcal{P}(A, B)$ and $\mathcal{Q}(F_0\, A, F_0\, B)$ have at most an arrow, the map $F_{A,B}$ exists iff

$$A \preceq B \quad \text{implies} \quad F_0\, A \preceq F_0\, B.$$

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$$F_0 : \mathrm{Obj}(\mathcal{P}) \to \mathrm{Obj}(\mathcal{Q})$$

$$\begin{aligned} F_1 &: \mathrm{Ar}(\mathcal{P}) \to \mathrm{Ar}(\mathcal{Q}) \\ F_{A,B} &: \mathcal{P}(A, B) \to \mathcal{Q}(F_0\,A, F_0\,B) \\ F_{A,B}\,f &: F_0\,A \to F_0\,B \end{aligned}$$

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That is, a functor $F : \mathcal{P} \to \mathcal{Q}$ is just a monotone map which sends, if exists, the unique arrow $A \to B$ to the unique arrow $F_0\,A \to F_0\,B$.

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Example from [Fong, Milewski and Spivak 2020, § 3.2.2].

Examples of Functors

Example

Let $(M, \cdot, \epsilon)$ and $(N, \diamond, \mu)$ be two monoids seen as categories, denoted $\mathcal{M}$ and $\mathcal{N}$, respectively. Let $*$ be the only object in both categories. A functor $F : \mathcal{M} \to \mathcal{N}$ is defined by

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$$F_0 * = *$$

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$$F_0 : \{*\} \to \{*\}$$
$$F_0 * = *$$

$$F_1 : \mathrm{Ar}(\mathcal{M}) \to \mathrm{Ar}(\mathcal{N})$$
$$F_{*,*} : \mathcal{P}(*, *) \to \mathcal{Q}(F_0 *, F_0 *)$$
$$F_{*,*}\, f : * \to *$$

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$$F_0 : \mathrm{Obj}(\mathcal{M}) \to \mathrm{Obj}(\mathcal{N}) \qquad F_1 : \mathrm{Ar}(\mathcal{M}) \to \mathrm{Ar}(\mathcal{N})$$

$$F_0 : \{*\} \to \{*\} \qquad F_{*,*} : \mathcal{P}(*,*) \to \mathcal{Q}(F_0\,*, F_0\,*)$$

$$F_0\,* = * \qquad F_{*,*}\,f : * \to *$$

The functor F must satisfies:

$$F_{*,*}\,(m_1 \cdot m_2) = (F_{*,*}\,m_1) \diamond (F_{*,*}\,m_2), \qquad \text{for all } m_1, m_2 \text{ in } \mathcal{M},$$

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$$F_0 * = * \qquad F_{*,*} f : * \to *$$

The functor F must satisfies:

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$$F_{*,*} \epsilon = \mu.$$

That is, a functor $F : \mathcal{M} \to \mathcal{N}$ is just a monoid homomorphism.

Examples of Functors

Example

The **identity functor** $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ in a category $\mathcal{C}$ is the functor that maps each object and each arrow of $\mathcal{C}$ to itself.

Examples of Functors

Example

Let $F : \mathbf{Mon} \to \mathbf{Set}$ be the **forgetful functor** which

(i) sends a monoid to its set of elements and

(ii) sends a homomorphism between monoids to the corresponding function between sets.

Examples of Functors

Example

Let $[S]$ be the set of all finite lists of elements of S. The **list functor**

$$\text{List} : \mathbf{Set} \to \mathbf{Set}, \qquad \text{is defined by}$$

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$$\mathsf{List}_0\, X := [X]$$

$$\mathsf{List}_1 : \mathrm{Ar}(\mathbf{Set}) \to \mathrm{Ar}(\mathbf{Set})$$

$$\mathsf{List}_{X,Y} : \mathbf{Set}(X, Y) \to \mathbf{Set}(\mathsf{List}_0\, X, \mathsf{List}_0\, Y)$$

$$\mathsf{List}_{X,Y}\, f : [X] \to [Y]$$

$$\mathsf{List}_{X,Y}\, f\, [x_1, x_2, \ldots, x_n] := [f(x_1), f(x_2), \ldots, f(x_n)]$$

Examples of Functors

Example

The **free monoid functor** MList $: \mathbf{Set} \to \mathbf{Mon}$ maps every set X to the free monoid over X.

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$$\begin{aligned} \mathsf{MList}_0 &: \mathrm{Obj}(\mathbf{Set}) \to \mathrm{Obj}(\mathbf{Mon}) \\ \mathsf{MList}_0\, X &:= (\mathsf{List}_0\, X, *, \varepsilon) \\ &= ([X], *, \varepsilon) \end{aligned}$$

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$$\begin{aligned} &\mathsf{MList}_1 : \mathrm{Ar}(\mathbf{Set}) \to \mathrm{Ar}(\mathbf{Mon}) \\ &\mathsf{MList}_{X,Y} : \mathbf{Set}(X, Y) \to \mathbf{Mon}(\mathsf{MList}_0\, X, \mathsf{MList}_0\, Y) \\ &\mathsf{MList}_{X,Y}\, f : ([X], *, \varepsilon) \to ([Y], *, \varepsilon) \\ &\mathsf{MList}_{X,Y}\, f\, [x_1, x_2, \ldots, x_n] := \mathsf{List}_{X,Y}\, f\, [x_1, x_2, \ldots, x_n] \end{aligned}$$

Exercises

Exercise 1

Verify that functors $F : \mathbf{2}_{\rightrightarrows} \to \mathbf{Set}$ correspond to directed graphs (textbook, Exercise 45).

Functors in Haskell

Functors in Haskell

Introduction via `Maybe`

(Whiteboard).

Functors in Haskell

Introduction via Maybe

(Whiteboard).

The typeclass Functor

```
class Functor f where
  fmap :: (a -> b) -> f a -> f b
```

Functors in Haskell

Example

The polymorphic type constructor Maybe is a functor whose instance is defined by

```
instance Functor Maybe where
  fmap _ Nothing  = Nothing
  fmap f (Just a) = Just (f a)
```

Functors in Haskell

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```

Exercise 2

Show that the Maybe functor satisfies the functoriality conditions.

Functors in Haskell

Example

ReadInt is a type constructor that turns any type a into a new type that reads a value of Int to create a value of a [Fong, Milewski and Spivak 2020, Example 3.41].

```
data ReadInt a = MkReadInt (Int -> a)
```

Functors in Haskell

Example

ReadInt is a type constructor that turns any type a into a new type that reads a value of Int to create a value of a [Fong, Milewski and Spivak 2020, Example 3.41].

```
data ReadInt a = MkReadInt (Int -> a)
```

ReadInt is a functor via the following instance.

```
instance Functor ReadInt where
  fmap f (MkReadInt g) = MkReadInt (f . g)
```

Functors in Haskell

Example

The (binary) function type `(->) :: a -> b -> (a -> b)` is a functor.

```
instance Functor ((->) a) where
  fmap f g = f . g
```

Note that `fmap :: (b -> c) -> (a -> b) -> (a -> c)`.

Functors in Haskell

Exercise 3

To define an instance of `Functor` for the (binary) product type `(,) :: a -> b -> (a,b)`.

Functors in Haskell

Example

Recall that terminal object (unit type) in Haskell is `()::()`. We can define a constant functor by

```
data CUnit a = MkCU ()

instance Functor CUnit where
  fmap f (MkCU ()) = MkCU ()
```

Functors in Haskell

Exercise 4

Given a constant 'functor' defined by

```
data CBool a = MkCB Bool

instance Functor CBool where
  fmap f (MkCB True)  = MkCB False
  fmap f (MkCB False) = MkCB True
```

Is `CBool` really a functor?

Functors in Haskell

Exercise 5

We define a constant functor by

```
data CInt a = MkCI Int
```

Show that the polymorphic type constructor `CInt` can be given the structure of a functor by saying how it lifts morphisms. That is, provide a Haskell function `mapCInt` of the type `(a -> b) -> (CInt a -> CInt b)` [Fong, Milewski and Spivak 2020, Exercise 3.46].

Functors in Haskell

Exercise 6

For each of the following type constructors, define two versions of `fmap`, one of which has a corresponding functor $\mathbf{Hask} \to \mathbf{Hask}$, and one of which does not [Fong, Milewski and Spivak 2020, Exercise 3.48].

(i) `data WithString a = WithStr (a, String)`

(ii) `data ConstStr a = ConstStr String`

(iii) `data List a = Nil | Cons (a, List a)`

Binary Functors

The Product Category

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. The **product category** $\mathcal{C} \times \mathcal{D}$ is defined by:

(i) Objects: (C, D), where C and D are objects in $\mathcal{C}$ and $\mathcal{D}$, respectively.

(ii) Arrows: $(C, D) \xrightarrow{(f,g)} (C', D')$, where $C \xrightarrow{f} C'$ and $D \xrightarrow{g} D'$ are arrows in $\mathcal{C}$ and $\mathcal{D}$, respectively.

(iii) Composition

$$(f', g') \circ (f, g) := (f' \circ f, g' \circ g).$$

(iv) Identities

$$\mathrm{id}_{(C,D)} := (\mathrm{id}_C, \mathrm{id}_D).$$

Definition of a Binary Functor

Definition

Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be three categories. A **binary functor** (or **bifunctor**) is a functor whose domain is a product category, that is, a binary functor from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ is a functor

$$F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}.$$

Example of Binary Functors

Example

The projection functors $\mathcal{C} \xleftarrow{\pi_1} \mathcal{C} \times \mathcal{D} \xrightarrow{\pi_2} \mathcal{D}$ are binary functors.

Example of Binary Functors

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The projection functors $\mathcal{C} \xleftarrow{\pi_1} \mathcal{C} \times \mathcal{D} \xrightarrow{\pi_2} \mathcal{D}$ are binary functors.

(i) For $\pi_1 : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ we have:

$$(\pi_1)_0 : \mathrm{Obj}(\mathcal{C} \times \mathcal{D}) \to \mathrm{Obj}(\mathcal{C})$$
$$(\pi_1)_0\,(C, D) := C$$

$$(\pi_1)_1 : \mathrm{Ar}(\mathcal{C} \times \mathcal{D}) \to \mathrm{Ar}(\mathcal{C})$$
$$(\pi_1)_{(C,D),(C',D')} : \mathrm{Mor}_{\mathcal{C}\times\mathcal{D}}((C, D), (C', D')) \to \mathrm{Mor}_{\mathcal{C}}((\pi_1)_0\,(C, D), (\pi_1)_0\,(C', D')),$$

$$(\pi_1)_{(C,D),(C',D')}\,(f, g) : C \to C'$$
$$(\pi_1)_{(C,D),(C',D')}\,(f, g) := f$$

Example of Binary Functors

Example

The projection functors $\mathcal{C} \xleftarrow{\pi_1} \mathcal{C} \times \mathcal{D} \xrightarrow{\pi_2} \mathcal{D}$ are binary functors.

(i) For $\pi_1 : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ we have:

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$$(\pi_1)_{(C,D),(C',D')}\,(f, g) : C \to C'$$
$$(\pi_1)_{(C,D),(C',D')}\,(f, g) := f$$

(ii) Similarly for $\pi_2 : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$.

The Product Functor

Definition

Let $\mathcal{C}$ be a category with binaries products, and let $\mathcal{C} \times \mathcal{C}$ be the product category of $\mathcal{C}$ with itself. The **product functor** $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a binary functor defined by

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$$\times_0 : \mathrm{Obj}(\mathcal{C} \times \mathcal{C}) \to \mathrm{Obj}(\mathcal{C})$$
$$\times_0 (A, B) := A \times B \quad \text{(binary product)}$$

$$\times_1 : \mathrm{Ar}(\mathcal{C} \times \mathcal{C}) \to \mathrm{Ar}(\mathcal{C})$$
$$\times_{(A,A'),(B,B')} : \mathrm{Mor}_{\mathcal{C}\times\mathcal{C}}((A, A'), (B, B')) \to \mathrm{Mor}_{\mathcal{C}}(\times_0 (A, A'), \times_0 (B, B'))$$
$$\times_{(A,A'),(B,B')} (f, g) : A \times A' \to B \times B'$$
$$\times_{(A,A'),(B,B')} (f, g) := f \times g \quad \text{(product morphish)}$$

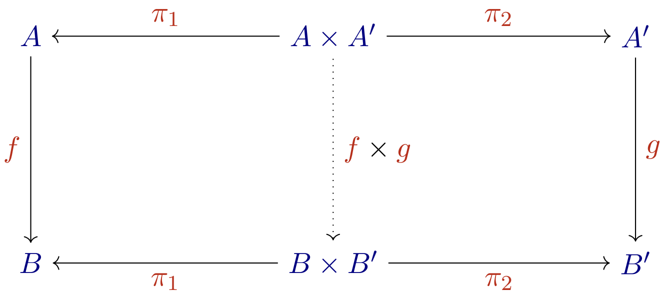
where $f \times g := \langle f \circ \pi_1, g \circ \pi_2 \rangle$.

(continued on next slide)

The Product Functor

Definition (continuation)

That is, both squares in the following diagram commute.



$$\begin{pmatrix} f \circ \pi_1 = \pi_1 \circ (f \times g) \\ g \circ \pi_2 = \pi_2 \circ (f \times g) \end{pmatrix}$$

N-Ary Functors

Remark

Binary functors can be generalised to n-ary functors.

Small, Large and Locally Small Categories

Small and Large Categories

Introduction

Before defining a category of categories, we need to classify the categories in small and large for avoiding that it be an object of itself.

Small and Large Categories

Definition

A category is **small** iff both the collection of its objects and the collection of its arrows are sets. Otherwise, the category is **large** [Awodey 2010].

Small and Large Categories

Example

The finite categories $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}$, a monoid viewed as a category, and a pre-order viewed as a category are small categories.

Small and Large Categories

Example

The finite categories $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}$, a monoid viewed as a category, and a pre-order viewed as a category are small categories.

Example

The categories $\mathbf{Set}$, $\mathbf{Pos}$, $\mathbf{Mon}$, $\mathbf{Grp}$ and $\mathbf{Top}$ are large categories.

Locally Small Categories

Definition

A category $\mathcal{C}$ is **locally small** iff for all objects A, B the collection $\mathcal{C}(A, B)$ is a set [Awodey 2010].

Locally Small Categories

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A category $\mathcal{C}$ is **locally small** iff for all objects A, B the collection $\mathcal{C}(A, B)$ is a set [Awodey 2010].

Remark

- Recall from the previous conventions that if the collection $\mathcal{C}(A, B)$ is a set it is called a hom-set and it is denoted $\hom_{\mathcal{C}}(A, B)$.
- Also recall that in the textbook all the collections $\mathcal{C}(A, B)$ are hom-sets.

Locally Small Categories

Example

Any small category is locally small.

Locally Small Categories

Example

Any small category is locally small.

Example

The categories $\mathbf{Set}$, $\mathbf{Pos}$, $\mathbf{Mon}$, $\mathbf{Grp}$ and $\mathbf{Top}$ are locally small categories.

The Category of Small Categories

The Category of Small Categories

Definition

The category $\mathbf{Cat}$ is the category of small categories:

(i) Objects: Small categories

(ii) Arrows: Functors

(continued on next slide)

The Category of Small Categories

Definition (continuation)

(iii) Composition of functors

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ two functors, then

$$
\begin{aligned}
G \circ F \quad &: \mathcal{C} \to \mathcal{E} \\
(G \circ F)_0 \quad &: \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{E}) \\
(G \circ F)_0\, A \quad &:= G_0\,(F_0\, A), \\
(G \circ F)_1 \quad &: \mathrm{Ar}(\mathcal{C}) \to \mathrm{Ar}(\mathcal{E}) \\
(G \circ F)_{A,B} \quad &: \mathcal{C}(A, B) \to \mathcal{E}((G \circ F)_0\, A, (G \circ F)_0\, B) \\
(G \circ F)_{A,B}\, f &: G_0\,(F_0\, A) \to G_0\,(F_0\, B) \\
(G \circ F)_{A,B}\, f &:= G_1\,(F_1\, f).
\end{aligned}
$$

(continued on next slide)

The Category of Small Categories

Definition (continuation)

(iii) Composition of functors

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, then

$$G \circ F : \mathcal{C} \to \mathcal{E} := \begin{cases} A & \mapsto G\,(F\,A), \\ f & \mapsto G\,(F\,f). \end{cases}$$

The Category of Small Categories

Definition (continuation)

(iii) Composition of functors

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, then

$$G \circ F : \mathcal{C} \to \mathcal{E} := \begin{cases} A & \mapsto G\,(F\,A), \\ f & \mapsto G\,(F\,f). \end{cases}$$

(iv) Identity functors

$$\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} := \begin{cases} A & \mapsto A, \\ f & \mapsto f. \end{cases}$$

The Category of Small Categories

Remark

The category $\mathbf{Cat}$ is large and therefore it is not object of itself.

Contravariance

Introduction

Description

A covariant functor F preserves the direction of arrows, that is,

$$F_1\,(f : A \to B) : F_0\,A \to F_0\,B.$$

A contravariant functor G reverses the direction of arrows, that is,

$$G_1\,(f : A \to B) : G_0\,B \to G_0\,A.$$

Contravariant Functors

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A **contravariant functor** G from $\mathcal{C}$ to $\mathcal{D}$ is a functor

$$G \; : \mathcal{C}^{\mathsf{op}} \to \mathcal{D} \text{ (or } \mathcal{C} \to \mathcal{D}^{\mathsf{op}}\text{)}$$

$$G_0 : \mathrm{Obj}(\mathcal{C}^{\mathsf{op}}) \to \mathrm{Obj}(\mathcal{D}) \qquad \text{(object-map)}$$

$$G_1 : \mathrm{Ar}(\mathcal{C}^{\mathsf{op}}) \to \mathrm{Ar}(\mathcal{D}) \qquad \text{(arrow-map)}$$

$$G_{A,B} \; : \mathcal{C}^{\mathsf{op}}(A, B) \to \mathcal{D}(G_0\, B, G_0\, A)$$

$$G_{A,B}\, f : G_0\, B \to G_0\, A$$

$$G_1(g \circ f) = (G_1\, f) \circ (G_1\, g) \qquad \text{(preservation of composition)}$$

$$G_1\,(\mathrm{id}_A) \; = \mathrm{id}_{(G_0\, A)} \qquad \text{(preservation of identities)}$$

Contravariant Functors

Example

Let $\mathcal{P}\, S$ be the power set of the set S. The **contravariant power set functor**

$$P^{\mathsf{op}} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Set}, \qquad \text{is defined by}$$

Contravariant Functors

Example

Let $\mathcal{P}\, S$ be the power set of the set S. The **contravariant power set functor**

$$P^{\mathsf{op}} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Set}, \qquad \text{is defined by}$$

$$P_0^{\mathsf{op}} : \mathrm{Obj}(\mathbf{Set}^{\mathsf{op}}) \to \mathrm{Obj}(\mathbf{Set})$$
$$P_0^{\mathsf{op}}\, X := \mathcal{P}\, X$$

Contravariant Functors

Example

Let $\mathcal{P}\,S$ be the power set of the set S. The **contravariant power set functor**

$$P^{\mathsf{op}} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Set}, \qquad \text{is defined by}$$

$$P_0^{\mathsf{op}} : \mathrm{Obj}(\mathbf{Set}^{\mathsf{op}}) \to \mathrm{Obj}(\mathbf{Set})$$
$$P_0^{\mathsf{op}}\,X := \mathcal{P}\,X$$

$$P_1^{\mathsf{op}} : \mathrm{Ar}(\mathbf{Set}^{\mathsf{op}}) \to \mathrm{Ar}(\mathbf{Set})$$
$${P^{\mathsf{op}}}_{X,Y} : \mathbf{Set}^{\mathsf{op}}(X, Y) \to \mathbf{Set}(P_0^{\mathsf{op}}\,Y, P_0^{\mathsf{op}}\,X)$$

$${P^{\mathsf{op}}}_{X,Y}\,f : \mathcal{P}\,Y \to \mathcal{P}\,X$$
$${P^{\mathsf{op}}}_{X,Y}\,f\,T := f^{-1}(T) = \{\, x \in X \mid f(x) \in T \,\}$$

Hom-Functors

Hom-Functors

Definition (first notation)

Let $\mathcal{C}$ be a locally small category and let A be an object of $\mathcal{C}$. The **covariant Set-valued hom-functor** $\mathcal{C}(A,-)$ is defined by

$$\mathcal{C}(A,-) : \mathcal{C} \to \mathbf{Set},$$

$$\mathcal{C}(A,-)_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathbf{Set})$$
$$\mathcal{C}(A,C)_0 := \mathcal{C}(A,C),$$

$$\mathcal{C}(A,-)_1 : \mathrm{Ar}(\mathcal{C}) \to \mathrm{Ar}(\mathbf{Set})$$
$$\mathcal{C}(A,-)_{C,D} : \mathcal{C}(C,D) \to \mathbf{Set}(\mathcal{C}(A,-)_0\, C, \mathcal{C}(A,-)_0\, D)$$
$$\mathcal{C}(A,f)_{C,D} : \mathcal{C}(A,C) \to \mathcal{C}(A,D)$$
$$\mathcal{C}(A,f)_{C,D}\, g := f \circ g.$$

Hom-Functors

Definition (first notation)

Let $\mathcal{C}$ be a (locally small) category and let B be an object of $\mathcal{C}$. The **contravariant** $\mathbf{Set}$**-valued hom-functor** $\mathcal{C}(-,B)$ is defined by

$$\mathcal{C}(-,B) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set},$$

$$\mathcal{C}(-,B)_0 : \mathrm{Obj}(\mathcal{C}^{\mathsf{op}}) \to \mathrm{Obj}(\mathbf{Set})$$
$$\mathcal{C}(C,B)_0 := \mathcal{C}(C,B),$$

$$\mathcal{C}(-,B)_1 : \mathrm{Ar}(\mathcal{C}^{\mathsf{op}}) \to \mathrm{Ar}(\mathbf{Set})$$
$$\mathcal{C}(-,B)_{C,D} : \mathcal{C}^{\mathsf{op}}(C,D) \to \mathbf{Set}(\mathcal{C}(-,B)_0\, D, \mathcal{C}(-,B)_0\, C)$$
$$\mathcal{C}(f,B)_{C,D} : \mathcal{C}(D,B) \to \mathcal{C}(C,B)$$
$$\mathcal{C}(f,B)_{C,D}\, g := g \circ f.$$

Hom-Functors

Exercise 7

Let $\mathcal{C}$ be a (locally small) category. Spell out the definition of the $\mathbf{set}$-valued hom-functor $\mathcal{C}(-,-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}$. Verify carefully that it is a functor (textbook, Exercise 47).

Hom-Functors

Notation

Recall that if $\mathcal{C}$ is a locally small category the collection of arrows of an object A to an object B is a set and it is denoted by $\hom_{\mathcal{C}}(A, B)$, that is,

$$\hom_{\mathcal{C}}(A, B) := \left\{ f \in \operatorname{Ar}(\mathcal{C}) \,\middle|\, A \xrightarrow{f} B \right\} =: \mathcal{C}(A, B).$$

Hom-Functors

Definition (second notation)

Let $\mathcal{C}$ be a locally small category and let A be an object of $\mathcal{C}$. The **covariant Set-valued hom-functor** $\hom_{\mathcal{C}}(A, -)$ is defined by

$$\hom_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathbf{Set},$$

$$\hom_{\mathcal{C}}(A, -)_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathbf{Set})$$
$$\hom_{\mathcal{C}}(A, C)_0 := \hom_{\mathcal{C}}(A, C)$$

$$\hom_{\mathcal{C}}(A, -)_1 : \mathrm{Ar}(\mathcal{C}) \to \mathrm{Ar}(\mathbf{Set})$$
$$\hom_{\mathcal{C}}(A, -)_{C,D} : \hom_{\mathcal{C}}(C, D) \to \mathbf{Set}(\hom_{\mathcal{C}}(A, C), \hom_{\mathcal{C}}(A, D))$$
$$\hom_{\mathcal{C}}(A, f : C \to D) : \hom_{\mathcal{C}}(A, C) \to \hom_{\mathcal{C}}(A, D)$$
$$\hom_{\mathcal{C}}(A, f : C \to D)\, g := f \circ g$$

Hom-Functors

Definition (second notation)

Let $\mathcal{C}$ be a (locally small) category and let B be an object of $\mathcal{C}$. The **contravariant** $\mathbf{Set}$**-valued hom-functor** $\hom_{\mathcal{C}}(-, B)$ is defined by

$$\hom_{\mathcal{C}}(-, B) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set},$$

$$\hom_{\mathcal{C}}(-, B)_0 : \mathrm{Obj}(\mathcal{C}^{\mathsf{op}}) \to \mathrm{Obj}(\mathbf{Set})$$
$$\hom_{\mathcal{C}}(C, B)_0 := \hom_{\mathcal{C}}(C, B)$$

$$\hom_{\mathcal{C}}(-, B)_1 : \mathrm{Ar}(\mathcal{C}^{\mathsf{op}}) \to \mathrm{Ar}(\mathbf{Set})$$
$$\hom_{\mathcal{C}}(-, B)_{C,D} : \hom_{(\mathcal{C}^{\mathsf{op}})}(C, D) \to \mathbf{Set}(\hom_{\mathcal{C}}(D, B), \hom_{\mathcal{C}}(C, B))$$

$$\hom_{\mathcal{C}}(f : C \to D, B) : \hom_{\mathcal{C}}(D, B) \to \hom_{\mathcal{C}}(C, B)$$
$$\hom_{\mathcal{C}}(f : C \to D, B)\, g := g \circ f$$

Hom-Functors

Exercise 8

Let $\mathcal{C}$ be a (locally small) category. Spell out the definition of the $\mathbf{set}$-valued hom-functor $\hom_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}$. Verify carefully that it is a functor (textbook, Exercise 47).

Properties of Functors

Faithful and Full Functors

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be (locally small) categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

(i) The functor F is **faithful** iff each map $F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(F_0\, A, F_0\, B)$ is injective.

(ii) The functor F is **full** iff each map $F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(F_0\, A, F_0\, B)$ is surjective.

Faithful and Full Functors

Example

The forgetful functor $F : \mathbf{Mon} \to \mathbf{Set}$ is faithful, but not full (explanation in the whiteboard).

Faithful and Full Functors

Example

The forgetful functor $F : \mathbf{Mon} \to \mathbf{Set}$ is faithful, but not full (explanation in the whiteboard).

Let $(M, \cdot, 1_M)$ and $(N, *, 1_N)$ be two monoids and let $f : M \to N$ be a homomorphism between them.

- Since $F_1\, f = f$, the map F_1 is injective.
- If $g : M \to N$ is any function in $\mathbf{Set}$ such that $g\,(1_M) \neq 1_N$, then g is not a homomorphism between $(M, \cdot, 1_M)$ and $(N, *, 1_N)$. Therefore the map F_1 is not surjective.

Faithful and Full Functors

Exercise 9

Show that the free monoid functor $\mathsf{MList} : \mathbf{Set} \to \mathbf{Mon}$ is faithful, but not full.

Exercise 10 (1.3.5.2)

Let $\mathcal{C}$ be a category with binary products such that, for each pair of objects A, B,

$$\mathcal{C}(A, B) \neq \emptyset. \tag{*}$$

(i) Show that the product functor $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is faithful.

(ii) Would $- \times -$ still be faithful in the absence of condition (*)?

Preservation and Reflection

Definition

Let P be a property of arrows and let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

(i) The functor F **preserves** the property P iff

$$\text{if } f \text{ satisfies } P \text{ then } F_1\, f \text{ satisfies } P.$$

(ii) The functor F **reflects** the property P iff

$$\text{if } F_1\, f \text{ satisfies } P \text{ then } f \text{ satisfies } P.$$

Preservation and Reflection

Example

Show that all functors preserve isomorphisms.

Preservation and Reflection

Example

Show that all functors preserve isomorphisms.

Example

Show that full and faithful functors reflect isomorphisms.

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