Category Theory and Functional Programming
Algebras and Recursive Data Types

Andrés Sicard-Ramírez

Universidad EAFIT

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, pages, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Abramsky and Tzevelekos 2011].

Outline

Introduction

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The relation between recursive (inductive) data types and initial algebras is described by [Fong, Milewski and Spivak 2020, p. 98] as:

> *The string of a recursive data type is a functor, and the knot is its initial algebra.*

Functors Associated with Recursive Data Types

Example (Natural numbers)

We define the recursive data type for natural numbers by

```
data Nat = Zero | Succ Nat
```

and we define the non-recursive associated functor by

```
data NatF a = ZeroF | SuccF a

instance Functor NatF where
  fmap :: (a -> b) -> NatF a -> NatF b
  fmap _ ZeroF      = ZeroF
  fmap f (SuccF n) = SuccF (f n)
```

Functors Associated with Recursive Data Types

Example (Lists)

We define the recursive data type for lists by

```
data List c = Nil | Cons c (List c)
```

and we define the non-recursive associated functor by

```
data ListF c a = NilF | ConsF c a

instance Functor (ListF c) where
  fmap :: (a -> b) -> ListF c a -> ListF c b
  fmap _ NilF          = NilF
  fmap f (ConsF x y) = ConsF x (f y)
```

F-Algebras

F-Algebras

Definition

Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor. An F-**algebra** (A, α) is [Fong, Milewski and Spivak 2020, p. 104]

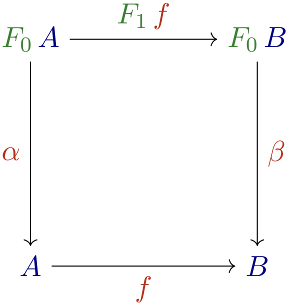
(i) an object A in $\mathrm{Obj}(\mathcal{C})$ (the **carrier**),

(ii) an arrow $\alpha : F_0\, A \to A$ (the **structure map**).

Algebra Arrows

Definition

Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor and let (A, α) and (B, β) be two F-algebras. An **algebra arrow** $f : (A, \alpha) \to (B, \beta)$ is an arrow $f : A \to B$ in $\mathcal{C}$ such that the following diagram commutes [Fong, Milewski and Spivak 2020, p. 104]:



$$\left(f \circ \alpha = \beta \circ (F_1\ f)\right)$$

References

References

- Abramsky, S. and Tzevelekos, N. (2011). Introduction to Categories and Categorical Logic. In: New Structures for Physics. Ed. by Coecke, B. Vol. 813. Lecture Notes in Physics. Springer, pp. 3–94. DOI: 10.1007/978-3-642-12821-9_1 (cit. on p. 2).
- Fong, B., Milewski, B. and Spivak, D. I. (2020). Programming with Categories (DRAFT). URL: http://brendanfong.com/programmingcats.html (cit. on pp. 5, 9, 10).