Automata and Formal Languages - CM0081
Proving Languages Not to Be Regular

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Semester 2018-2
Properties of Regular Languages

- Proving languages not to be regular
- Closure properties
- Decision properties
- Equivalence and minimization of automata
Introduction

- Is \( L = \{0^m 1^n \mid m, n \geq 0\} \) a regular language?

Yes! \( L = L(0^* 1^*) \).

Yes! \( L = L(0^+ 1^+) \).

Yes! \( L = L(000^* 11111^*) \).

No! Informal proof (whiteboard).
The Pumping Lemma

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- Is $L_{01} = \{0^n1^n \mid n \geq 1\}$ a regular language?
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  No! Informal proof (whiteboard).
The Pumping Lemma

Theorem (Pumping Lemma for regular languages, 4.1)

Let $L$ be a regular language, then

$$(\exists n \in \mathbb{Z}^+)(\forall w \in L)(|w| \geq n \Rightarrow (\exists x \exists y \exists z)(w = xyz))$$

such that

1. $y \neq \varepsilon$,
2. $|xy| \leq n$ and
3. $(\forall k \geq 0) \ xy^kz \in L$. 

Proving Languages Not to Be Regular
The Pumping Lemma

Proof.

1. Suppose \( L \) is a regular language. Exist a DFA \( A = (Q, \Sigma, \delta, q_0, F) \) with \( n \) states such that \( L(A) = L \).
The Pumping Lemma

Proof.

1. Suppose $L$ is a regular language. Exist a DFA $A = (Q, \Sigma, \delta, q_0, F)$ with $n$ states such that $L(A) = L$.

2. Let $w = a_1 \ldots a_m \in L$, $m \geq n$ and $q_i = \hat{\delta}(q_0, a_1 \ldots a_i)$. 

-Proving Languages Not to Be Regular-
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2. Let $w = a_1 \ldots a_m \in L$, $m \geq n$ and $q_i = \hat{\delta}(q_0, a_1 \ldots a_i)$.

3. By the pigeonhole principle, exists $i$ and $j$, with $0 \leq i < j \leq n$ such that $q_i = q_j$.

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3. By the pigeonhole principle, exists $i$ and $j$, with $0 \leq i < j \leq n$ such that $q_i = q_j$.

4. Let $w = xyz$ where

$$y = a_{i+1} \ldots a_j$$

$$x = a_1 \ldots a_i$$

$$z = a_{j+1} \ldots a_m$$
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1. Suppose $L$ is a regular language. Exist a DFA $A = (Q, \Sigma, \delta, q_0, F)$ with $n$ states such that $L(A) = L$.

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4. Let $w = xyz$ where

\[ y = a_{i+1} \ldots a_j \]

\[ x = a_1 \ldots a_i \]

\[ z = a_{j+1} \ldots a_m \]

5. Then $(\forall k \geq 0) \; xyz^k \in L$. 

\[ 
\begin{align*}
\text{start} & \rightarrow q_0 \\
& \quad \xrightarrow{x = a_1 \ldots a_i} q_i \\
& \quad \xrightarrow{y = a_{i+1} \ldots a_j} q_i \\
& \quad \xrightarrow{z = a_{j+1} \ldots a_m} q_f
\end{align*} \]
Application of the Pumping Lemma: Proving Languages Not to Be Regular

Method
Whiteboard
Method
Whiteboard

Exercise (4.1.2.e)
Let $\Sigma = \{0, 1\}$ be an alphabet and let $L = \{ww \mid w \in \Sigma^*\}$ be the so-called copy language. Prove that $L$ is not regular.
Exercise (cont.)

Proof.

1. Suppose $L$ is regular.
Exercise (cont.)

Proof.

1. Suppose $L$ is regular.
2. Let $n \in \mathbb{Z}^+$ be a constant (according to the Pumping Lemma).
Proof.

1. Suppose $L$ is regular.
2. Let $n \in \mathbb{Z}^+$ be a constant (according to the Pumping Lemma).
3. Let $w = 0^n1^n1 \in L$ and $|w| \geq n$. 

Exercise (cont.)
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Proof.

1. Suppose $L$ is regular.
2. Let $n \in \mathbb{Z}^+$ be a constant (according to the Pumping Lemma).
3. Let $w = 0^n10^n1 \in L$ and $|w| \geq n$.
4. For the Pumping Lemma exists $x, y$ and $z$ such that $w = xyz$, $|xy| \leq n$, $y \neq \varepsilon$ and $\forall k \geq 0. xy^kz \in L$. 
Proof.

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5. Then $y = 0^m$, $0 < m \leq n$. 

Exercise (cont.)
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1. Suppose \( L \) is regular.
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3. Let \( w = 0^n10^n1 \in L \) and \( |w| \geq n \).
4. For the Pumping Lemma exists \( x, y \) and \( z \) such that \( w = xyz \), \( |xy| \leq n \), \( y \neq \varepsilon \) and \( \forall k \geq 0. \ xy^kz \in L \).
5. Then \( y = 0^m \), \( 0 < m \leq n \).
6. But, \( xy^0z \notin L \) which is a contradiction by the Pumping Lemma.
Exercise (cont.)

Proof.

1. Suppose \( L \) is regular.
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5. Then \( y = 0^m \), \( 0 < m \leq n \).
6. But, \( xy^0z \notin L \) which is a contradiction by the Pumping Lemma.
7. Therefore, \( L \) is not regular.
Exercise (4.1.2.a)
Let $L$ be the language

$$L = \{0^n \mid n \text{ is a perfect square}\}.$$

Prove that $L$ is not regular.
Exercise (cont.)

Proof.

1. Suppose $L$ is regular.
Application of the Pumping Lemma: Proving Languages Not to Be Regular

Exercise (cont.)

Proof.

1. Suppose $L$ is regular.

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Exercise (cont.)

Proof.

1. Suppose $L$ is regular.

2. Let $n \in \mathbb{Z}^+$ be a constant (according to the Pumping Lemma).

3. Let $w = 0^{n^2} \in L$ and $|w| \geq n$. 
Exercise (cont.)

Proof.

1. Suppose \( L \) is regular.

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4. For the Pumping Lemma exists \( x, y \) and \( z \) such that \( w = xyz \), \( |xy| \leq n \), \( y \neq \varepsilon \) and \( \forall k \geq 0. xy^kz \in L \).
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Proof.

1. Suppose $L$ is regular.

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4. For the Pumping Lemma exists $x, y$ and $z$ such that $w = xyz$, $|xy| \leq n$, $y \neq \varepsilon$ and $\forall k \geq 0. \ xy^kz \in L$.

5. Then $y = 0^m$, $0 < m \leq n$ and $n^2 + 1 \leq |xyyz| \leq n^2 + n$. 
Exercise (cont.)

Proof.

1. Suppose $L$ is regular.
2. Let $n \in \mathbb{Z}^+$ be a constant (according to the Pumping Lemma).
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4. For the Pumping Lemma exists $x, y$ and $z$ such that $w = xyz$, $|xy| \leq n$, $y \neq \varepsilon$ and $\forall k \geq 0. \ xy^kz \in L$.
5. Then $y = 0^m$, $0 < m \leq n$ and $n^2 + 1 \leq |xyyz| \leq n^2 + n$.
6. Since the next perfect square after $n^2$ is $(n + 1)^2 = n^2 + 2n + 1$, we know that $xyyz \notin L$ ($|xyyz|$ is strictly between the consecutive perfect squares $n^2$ and $(n + 1)^2$).
Application of the Pumping Lemma: Proving Languages Not to Be Regular

Exercise (cont.)

Proof.

1. Suppose \( L \) is regular.

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4. For the Pumping Lemma exists \( x, y \) and \( z \) such that \( w = xyz \), \( |xy| \leq n \), \( y \neq \varepsilon \) and \( \forall k \geq 0. \ xy^kz \in L \).

5. Then \( y = 0^m \), \( 0 < m \leq n \) and \( n^2 + 1 \leq |xyyz| \leq n^2 + n \).

6. Since the next perfect square after \( n^2 \) is \( (n + 1)^2 = n^2 + 2n + 1 \), we know that \( xy^yz \not\in L \) (\(|xyyz|\) is strictly between the consecutive perfect squares \( n^2 \) and \( (n + 1)^2 \)).

7. This a contradiction by the Pumping Lemma.
Application of the Pumping Lemma: Proving Languages Not to Be Regular

Exercise (cont.)

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7. This a contradiction by the Pumping Lemma.
8. Therefore, \( L \) is not regular.
Remark

Frishberg and Gasarch [2018] show other methods and some open problems when proving that a language is not regular.