Automata and Formal Languages - CM0081
Closure Properties of Regular Languages

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Introduction

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- **Intersection**: $L \cap M$
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- Difference: $L - M$
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- Reversal: $L^R = \{w^r \in \Sigma^* \mid w \in L\}$
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- **Closure:** $L^*$
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- Concatenation: $LM$
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- **Homomorphism**: $h(L) = \{h(w) \mid w \in L \text{ and } h \text{ is a homomorphism}\}$
Let $L$ and $M$ be regular languages. The following languages are regular:

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- **Concatenation:** $LM$
- **Homomorphism:** $h(L) = \{h(w) \mid w \in L \text{ and } h \text{ is a homomorphism}\}$
- **Inverse homomorphism:**
  $$h^{-1}(L) = \{w \in \Sigma \mid h(w) \in L \text{ and } h \text{ is a homomorphism}\}$$
Theorem (4.4)

If $L$ and $M$ are regular languages, then so is $L \cup M$. 
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Proof
(Using regular expressions)
Theorem (4.5)

Let $\overline{L} = \Sigma^* - L$ be the complement of a language $L$. If $L$ is a regular language, then so is $\overline{L}$.
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Proof.
Let

$$A = (Q, \Sigma, \delta, q_0, F)$$

be a DFA such that $L(A) = L$. Then

$$B = (Q, \Sigma, \delta, q_0, Q - F)$$

is a DFA such that $L(B) = \overline{L}$. 

\[\square\]
Question

“Do you see how to take a regular expression and change it into one that defines the complement language?” [Hopcroft, Motwani and Ullman 2007, p. 136]
Closure Under Union and Complementation

Using the closure properties we can prove that a language is not regular.

Example

Given that $L = \{w \in \{0, 1\}^* \mid w$ has an equal numbers of 0's and 1's $\}$ is a language not regular. Prove that $L' = \{w \in \{0, 1\}^* \mid w$ has an unequal numbers of 0's and 1's $\}$ is a language not regular.

Proof

Whiteboard.
Using the closure properties we can prove that a language is not regular.

Example

Given that

\[ L = \{ w \in \{0, 1\}^* \mid w \text{ has an equal numbers of 0's and 1's} \} \]

is a language not regular. Prove that

\[ L' = \{ w \in \{0, 1\}^* \mid w \text{ has an unequal numbers of 0's and 1's} \} \]

is a language not regular.
Closure Under Union and Complementation

Using the closure properties we can prove that a language is not regular.

Example

Given that

\[ L_\leq = \{ w \in \{0, 1\}^* \mid w \text{ has an equal numbers of } 0\text{'s and } 1\text{'s} \} \]

is a language not regular. Prove that

\[ L_\neq = \{ w \in \{0, 1\}^* \mid w \text{ has an unequal numbers of } 0\text{'s and } 1\text{'s} \} \]

is a language not regular.

Proof

Whiteboard.
Let $A_L$, $A_M$ and $A$ be DFAs given by

\[
A_L = (Q_L, \Sigma, \delta_L, q_L, F_L),
\]
\[
A_M = (Q_M, \Sigma, \delta_M, q_M, F_M),
\]
\[
A = (Q_L \times Q_M, \Sigma, \delta, (q_L, q_M), F_L \times F_M),
\]

where

\[
\delta : (Q_L \times Q_M) \times \Sigma \rightarrow Q_L \times Q_M
\]
\[
\delta((p, q), a) = (\delta_L(p, a), \delta_M(q, a)).
\]
Product Construction

Let $A_L$, $A_M$ and $A$ be DFAs given by

$$A_L = (Q_L, \Sigma, \delta_L, q_L, F_L),$$
$$A_M = (Q_M, \Sigma, \delta_M, q_M, F_M),$$
$$A = (Q_L \times Q_M, \Sigma, \delta, (q_L, q_M), F_L \times F_M),$$

where

$$\delta : (Q_L \times Q_M) \times \Sigma \rightarrow Q_L \times Q_M$$
$$\delta((p, q), a) = (\delta_L(p, a), \delta_M(q, a)).$$

Theorem (Exercise 4.2.15)

For all $w \in \Sigma^*$,

$$\hat{\delta}((q_L, q_M), w) = (\hat{\delta}_L(q_L, w), \hat{\delta}_M(q_M, w)).$$
Product Construction

Proof by induction on $w$.

1. Basis step

$$\hat{\delta}((q_L, q_M), \varepsilon) = (q_L, q_M)$$

$$= (\hat{\delta}_L(q_L, \varepsilon), \hat{\delta}_M(q_M, \varepsilon))$$

(def. of $\hat{\delta}$)

(def. of $\hat{\delta}_L$ and $\hat{\delta}_M$)
Proof by induction on $w$.  

1. Basis step

$$\hat{\delta}((q_L, q_M), \varepsilon) = (q_L, q_M)$$  

(def. of $\hat{\delta}$)

$$= (\hat{\delta}_L(q_L, \varepsilon), \hat{\delta}_M(q_M, \varepsilon))$$  

(def. of $\hat{\delta}_L$ and $\hat{\delta}_M$)

2. Inductive step

$$\hat{\delta}((q_L, q_M), xa)$$

$$= \delta(\hat{\delta}((q_L, q_M), x), a)$$  

(def. of $\hat{\delta}$)

$$= \delta((\hat{\delta}_L(q_L, x), \hat{\delta}_M(q_M, x)), a)$$  

(by IH)

$$= (\delta_L(\hat{\delta}_L(q_L, x), a), \delta_M(\hat{\delta}_M(q_M, x), a))$$  

(def. of $\delta$)

$$= (\hat{\delta}_L(q_L, xa), \hat{\delta}_M(q_M, xa))$$  

(def. of $\hat{\delta}_L$ and $\hat{\delta}_L$)
Theorem (4.8)

If $L$ and $M$ are regular languages, then so is $L \cap M$. 

Closure Under Intersection

Theorem (4.8)
If $L$ and $M$ are regular languages, then so is $L \cap M$.

Proof.
Let $A_L$ and $A_M$ be DFAs accepting $L$ and $M$. The product construction of $A_L$ and $A_M$ accepts $L \cap M$. 

Different proof.
The regular languages are closure under union and complement, and $L \cap M = L \cup M$. 

Closure Properties of Regular Languages
Closure Under Intersection

Theorem (4.8)
If $L$ and $M$ are regular languages, then so is $L \cap M$.

Proof.
Let $A_L$ and $A_M$ be DFAs accepting $L$ and $M$. The product construction of $A_L$ and $A_M$ accepts $L \cap M$.

Different proof.
The regular languages are closure under union and complement, and

$$L \cap M = \overline{L \cup M}.$$
**Closure Under Reversal**

**Definition**

Let $w = a_1 a_2 \cdots a_n$ be a word. The **reversal** of $w$ is defined by

$$w^R = a_n a_{n-1} \cdots a_1.$$
Closure Under Reversal

Definition
Let $w = a_1 a_2 \cdots a_n$ be a word. The reversal of $w$ is defined by

$$w^R = a_n a_{n-1} \cdots a_1.$$ 

Definition
Let $L$ be a language on alphabet $\Sigma$. The reversal of $L$ is defined by

$$L^R = \{w^R \in \Sigma^* \mid w \in L\}.$$
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Definition
Let $L$ be a language on alphabet $\Sigma$. The reversal of $L$ is defined by

$$L^R = \{w^R \in \Sigma^* \mid w \in L\}.$$  

Theorem (4.11)
If $L$ is regular language, then so is $L^R$ (proof using automata or regular expressions)
Closure Under Reversal

Proof using automata.

Let $L$ be recognized by a finite automaton $A$. Turn $A$ into finite automaton for $L^R$, by

1. Reversing all arcs.
2. Make the start state of $A$ be the only accepting state.
3. Create a new start state $p_0$ with transitions $\delta(p_0, \varepsilon) = f$, where $f \in F$ are the accepting states of $A$. 

\[
\begin{align*}
\delta(p_0, \varepsilon) &= f, \\
&= \text{accepting states of } A.
\end{align*}
\]
Closure Under Reversal

Example

A NFA accepting all the binary strings that end in 01.

\[
\begin{align*}
q_0 & \xrightarrow{0,1} q_1 \\
q_1 & \xrightarrow{1} q_2
\end{align*}
\]
Closure Under Reversal

Example

A NFA accepting all the binary strings that end in 01.

A NFA accepting all the binary strings that start with 10.
Homomorphisms

Definition
An algebraic structure on a set $A \neq \emptyset$ is essentially a collection of $n$-ary operations on $A$ [Birkhoff 1946, 1987].
Homomorphisms

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An **algebraic structure** on a set $A \neq \emptyset$ is essentially a collection of $n$-ary operations on $A$ [Birkhoff 1946, 1987].

Example (Semigroup)

A **semigroup** $(S, \ast)$ is a set $S$ with an associative binary operation $\ast : S \times S \to S$. 

Homomorphisms

Definition

An **algebraic structure** on a set $A \neq \emptyset$ is essentially a collection of $n$-ary operations on $A$ [Birkhoff 1946, 1987].

Example (Semigroup)

A **semigroup** $(S, \ast)$ is a set $S$ with an associative binary operation $\ast : S \times S \to S$.

Example (Monoid)

A **monoid** $(M, \ast, \varepsilon)$ is a semigroup $(M, \ast)$ with an element $\varepsilon \in M$ which is an unit for $\ast$, i.e. $\forall x. x \ast \varepsilon = \varepsilon \ast x = x$. 
Homomorphisms

Definition

A homomorphism is a structure-preserving map between two algebraic structures.
Homomorphisms

Definition

A **homomorphism** is a structure-preserving map between two algebraic structures.

Example

A homomorphism between two semigroups \((S, \ast)\) and \((S', \ast')\) is a function \(\varphi : S \rightarrow S'\) such that:

\[
\forall x \ y. \varphi(x \ast y) = \varphi(x) \ast' \varphi(y).
\]

Graphically (see whiteboard).
Homomorphisms

Example

A homomorphism between two monoids \((M, *, \varepsilon)\) and \((M', *, \varepsilon')\) is a function \(\varphi : M \rightarrow M'\) such that:

\[
\forall x \ y. \varphi(x * y) = \varphi(x) *' \varphi(y),
\]

\[
\varphi(\varepsilon) = \varepsilon'.
\]
Homomorphisms

Definition
A homomorphism $\varphi$ between two algebraic structures is [Cohn 1981]:

- a **monomorphism** if $\varphi$ is an injection,
- an **epimorphism** if $\varphi$ is a surjection,
- an **endomorphism** if $\varphi$ is from an algebraic structure to itself,
- an **isomorphism** if $\varphi$ is a bijection,
- an **automorphism** if $\varphi$ is a bijective endomorphism.
Closure Under Homomorphism

Definition

Let $\Sigma$ and $\Gamma$ be two alphabets. A **homomorphism** between $(\Sigma^*, \cdot, \varepsilon)$ and $(\Gamma^*, \cdot, \varepsilon)$ is a function $h : \Sigma^* \rightarrow \Gamma^*$

$$a_1 a_2 \cdots a_n \mapsto h(a_1)h(a_2)\cdots h(a_n)$$

$$\varepsilon \mapsto \varepsilon$$

Note: For this reason the textbook talk about a homomorphism $h : \Sigma \rightarrow \Gamma^*$. 
Example

Let \( h : \{0, 1\}^* \rightarrow \{a, b\}^* \) be a homomorphism defined by

\[
\begin{align*}
h(0) &= ab, \\
h(1) &= \varepsilon.
\end{align*}
\]

Then

\[
\begin{align*}
h(0011) &= h(0)h(0)h(1)h(1) \\
&= abab.
\end{align*}
\]
Closure Under Homomorphism

Definition

Let $L$ be a language over an alphabet $\Sigma$ and let $h$ be a homomorphism on $\Sigma$. The application of $h$ to $L$, denoted $h(L)$, is defined by†

$$h(L) = \{ h(w) \mid w \in L \}.$$ 

†Figure from Hopcroft, Motwani and Ullman [2007, Fig. 4.5a].
Example

Let \( h : \{0, 1\}^* \rightarrow \{a, b\}^* \) be a homomorphism defined by

\[
    h(0) = ab,
    h(1) = \varepsilon.
\]

If \( L = L(10^*1) \), then \( h(L) = L((ab)^*) \).
Closure Under Homomorphism

Theorem (4.14)
If $L$ is a regular language over the alphabet $\Sigma$ and $h$ is a homomorphism on $\Sigma$, then $h(L)$ is also regular.
Closure Under Homomorphism

Theorem (4.14)
If \( L \) is a regular language over the alphabet \( \Sigma \) and \( h \) is a homomorphism on \( \Sigma \), then \( h(L) \) is also regular.

Proof plan
- Let \( E \) be a regular expression such that \( L = L(E) \).
- Let \( h(E) \) be the regular expression replacing each symbol \( a \in \Sigma \) by \( h(a) \) in the regular expression \( E \).
- We need to prove that \( L(h(E)) = h(L(E)) \).
Closure Under Homomorphism

Proving $L(h(E)) = h(L(E))$.

Basis step

- $E$ is $\varepsilon$ or $\emptyset$.
  1. $h(E) = E$ ($h$ does not affect $E$)
  2. $h(L(E)) = L(E)$ ($L(E)$ is empty or only contains $\varepsilon$)
  3. $L(h(E)) = L(E) = h(L(E))$ (by 1 and 2)
Closure Under Homomorphism

Proving $L(h(E)) = h(L(E))$.

Basis step

- $E$ is $\varepsilon$ or $\emptyset$.
  1. $h(E) = E$ ($h$ does not affect $E$)
  2. $h(L(E)) = L(E)$ ($L(E)$ is empty or only contains $\varepsilon$)
  3. $L(h(E)) = L(E) = h(L(E))$ (by 1 and 2)

- $E = a$
  1. $L(E) = \{a\}$
  2. $h(L(E)) = \{h(a)\}$
  3. $h(E)$ is the regular expression that is the string of symbols $h(a)$
  4. $L(h(E)) = \{h(a)\}$
  5. $L(h(E)) = h(L(E))$ (by transitivity between 2 and 4)
Closure Under Homomorphism

Proving $L(h(E)) = h(L(E))$.

Inductive step

1. $L(E) = L(F) ∪ L(G)$ (def. of $+$)
2. $h(E) = h(F + G) = h(F) + h(G)$ (def. of $h(E)$)
3. $L(h(E)) = L(h(F) + h(G)) = L(h(F)) ∪ L(h(G))$ (def. of $+$)
4. $h(L(E)) = h(L(F) ∪ L(G)) = h(L(F)) ∪ h(L(G))$ ($h$ is applied to a language by application to each of its strings)
5. $L(h(F)) = h(L(F))$ and $L(h(G)) = h(L(G))$ (IH)
6. $L(h(E)) = h(L(E))$
Proving $L(h(E)) = h(L(E))$.

Inductive step (cont.)

- $E = FG$ (similar to the previous case)
Closure Under Homomorphism

Proving $L(h(E)) = h(L(E))$.

Inductive step (cont.)

- $E = FG$ (similar to the previous case)
- $E = F^*$ (similar to the previous case)
  1. $L(E) = (L(F))^*$ (def. of $*$)
  2. $h(E) = h(F^*) = (h(F))^*$ (def. of $h(E)$)
  3. $L(h(E)) = L((h(F))^*) = (L(h(F)))^*$ (def. of $*$)
  4. $h(L(E)) = h((L(F))^*) = (h(L(F)))^*$ ($h$ is applied to a language by application to each of its strings)
  5. $L(h(F)) = h(L(F))$ (IH)
  6. $L(h(E)) = h(L(E))$
Example

Prove that

$$L = \{0^i1^j2^k \mid i, j, k \in \mathbb{Z}^+ \text{ and } i \neq j \neq k\}$$

is a language not regular.
Example

Prove that

\[ L = \{0^i1^j2^k \mid i, j, k \in \mathbb{Z}^+ \text{ and } i \neq j \neq k \} \]

is a language not regular.

Proof.

1. We define the homomorphism

\[ h(0) = 0, \quad h(1) = 1, \quad h(2) = \varepsilon. \]
Example

Prove that

\[ L = \{0^i 1^j 2^k \mid i, j, k \in \mathbb{Z}^+ \text{ and } i \neq j \neq k \} \]

is a language not regular.

Proof.

1. We define the homomorphism

   \[ h(0) = 0, \quad h(1) = 1, \quad h(2) = \varepsilon. \]

2. The homomorphism \( h \) removes the \( 2^k \)'s, so

   \[ h(L) = \{0^i 1^j \mid i, j \in \mathbb{Z}^+ \text{ and } i \neq j \}. \]
Example

Prove that

\[ L = \{0^i1^j2^k \mid i, j, k \in \mathbb{Z}^+ \text{ and } i \neq j \neq k\} \]

is a language not regular.

Proof.

1. We define the homomorphism

   \[ h(0) = 0, \quad h(1) = 1, \quad h(2) = \varepsilon. \]

2. The homomorphism \( h \) removes the \( 2^k \)'s, so

   \[ h(L) = \{0^i1^j \mid i, j \in \mathbb{Z}^+ \text{ and } i \neq j\}. \]

3. We know that \( h(L) \) is not regular, so \( L \) is not regular.
Example

Let $L$ be a regular language and $h$ a homomorphism on $L$. Define $h^*(L)$ by

$$h^*(L) = L \cup h(L) \cup h(h(L)) \cup h(h(h(L))) \cup ...$$

Is $h^*(L)$ necessarily regular?

†From somewhere in Internet (I don’t remember).
Example

Let $L$ be a regular language and $h$ a homomorphism on $L$. Define $h^*(L)$ by

$$h^*(L) = L \cup h(L) \cup h(h(L)) \cup h(h(h(L))) \cup \ldots$$

Is $h^*(L)$ necessarily regular?

No. Let $L = \{01\}$ and $h$ defined as $h(0) = 00$ and $h(1) = 11$. Then

$$h^*(L) = \{01, 0011, 00001111, \ldots \}$$

$$= \{0^n1^n \mid n = 2^k \text{ for } k \geq 0\},$$

which is a language not regular.†

†From somewhere in Internet (I don’t remember).
Closure Under Inverse Homomorphism

Definition
Let \( h : \Sigma^* \rightarrow \Gamma^* \) be a homomorphism and \( L \subseteq \Gamma^* \) a language. The application of \( h^{-1} \) to \( L \), denoted \( h^{-1}(L) \), is defined by†

\[
h^{-1}(L) = \{ w \in \Sigma^* \mid h(w) \in L \}.
\]

Remark
Note that \( h^{-1} \) is a relation but it is not necessarily a function.

†Figure from Hopcroft, Motwani and Ullman [2007, Fig. 4.5b].
Example

Let $h : \{a, b\} \rightarrow \{0, 1\}^*$ a homomorphism defined by

$$h(a) = 01, \quad h(b) = 10.$$ 

If $L = L((00 + 1)^*)$, then $h^{-1}(L) = L((ba)^*)$

Note that $h^{-1}$ is not a function, but a relation.

(It is necessary to prove $h(w) \in L \iff w = baba \cdots ba$).
Closure Under Inverse Homomorphism

Theorem (4.16)

Let \( h : \Sigma^* \to \Gamma^* \) be a homomorphism and \( L \subseteq \Gamma^* \) a regular language. Then \( h^{-1}(L) \) is regular (proof using automata).
Example

Prove that $L = \{0^n 1^{2n} \mid n \geq 0\}$ is a language not regular.
Example

Prove that \( L = \{0^n1^{2n} \mid n \geq 0\} \) is a language not regular.

Proof.

1. Given the homomorphism

\[
\begin{align*}
  h(0) &= 0, \\
  h(1) &= 11,
\end{align*}
\]

then \( h^{-1}(L) = \{0^n1^n \mid n \geq 0\} \).
Example

Prove that \( L = \{0^n1^{2n} \mid n \geq 0\} \) is a language not regular.

Proof.

1. Given the homomorphism

\[
\begin{align*}
    h(0) &= 0, \\
    h(1) &= 11,
\end{align*}
\]

then \( h^{-1}(L) = \{0^n1^n \mid n \geq 0\} \).

2. Since \( h^{-1}(L) \) is not regular, then \( L \) is not regular.
Exercise (4.2.2)

If $L$ is a language, and $a$ is a symbol, then $L/a$, the quotient of $L$ and $a$, is the set of strings $w$ such that $wa$ is in $L$. For example, if $L = \{a, aab, baa\}$, then $L/a = \{\varepsilon, ba\}$. Prove that if $L$ is regular, so is $L/a$. Hint: Start with a DFA for $L$ and consider the set of accepting states.
Exercise (4.2.2)

If \( L \) is a language, and \( a \) is a symbol, then \( L/a \), the quotient of \( L \) and \( a \), is the set of strings \( w \) such that \( wa \) is in \( L \). For example, if \( L = \{ a, aab, baa \} \), then \( L/a = \{ \varepsilon, ba \} \). Prove that if \( L \) is regular, so is \( L/a \). Hint: Start with a DFA for \( L \) and consider the set of accepting states.

Proof (Hopcroft, Motwani and Ullman [2007] solution).

Start with a DFA \( A \) for \( L \). Construct a new DFA \( B \), that is exactly the same as \( A \), except that state \( q \) is an accepting state of \( B \) if and only if \( \delta(q, a) \) is an accepting state of \( A \). Then \( B \) accepts input string \( w \) if and only if \( A \) accepts \( wa \); that is, \( L(B) = L/a \).
Exercise (4.2.3)

If \( L \) is a language, and \( a \) is a symbol, then \( a \backslash L \) is the set of strings \( w \) such that \( aw \) is in \( L \). For example, if \( L = \{a, aab, baa\} \), then \( a \backslash L = \{\varepsilon, ab\} \). Prove that if \( L \) is regular, so is \( a \backslash L \). Hint: Start with a DFA for \( L \) and consider its start state.
Exercise (4.2.3)

If $L$ is a language, and $a$ is a symbol, then $a \setminus L$ is the set of strings $w$ such that $aw$ is in $L$. For example, if $L = \{a, aab, baa\}$, then $a \setminus L = \{\varepsilon, ab\}$. Prove that if $L$ is regular, so is $a \setminus L$. Hint: Start with a DFA for $L$ and consider its start state.

Proof (Hopcroft, Motwani and Ullman [2007] solution).

Start with a DFA $A$ for $L$. Construct a new DFA $B$, that is exactly the same as $A$, except that its start state is $\delta(q_0, a)$ where $q_0$ is the start state of $A$. Then $B$ accepts input string $w$ if and only if $A$ accepts $aw$; that is, $L(B) = L \setminus a$. 

\hfill $\blacksquare$

