Let $L$ and $M$ be regular languages. The following languages are regular:

- **Union**: $L \cup M$
- **Intersection**: $L \cap M$
- **Complement**: $L^c$
- **Difference**: $L - M$
- **Reversal**: $L^R = \{ w^r \in \Sigma^* \mid w \in L \}$
- **Closure**: $L^*$
- **Concatenation**: $LM$
- **Homomorphism**: $h(L) = \{ h(w) \mid w \in L \text{ and } h \text{ is a homomorphism} \}$
- **Inverse homomorphism**: $h^{-1}(L) = \{ w \in \Sigma \mid h(w) \in L \text{ and } h \text{ is a homomorphism} \}$
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Introduction

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Closure Under Union and Complementation

Theorem (4.4)

If $L$ and $M$ are regular languages, then so is $L \cup M$. 
Theorem (4.4)
If $L$ and $M$ are regular languages, then so is $L \cup M$.

Proof
(Using regular expressions)
Theorem (4.5)

Let $\overline{L} = \Sigma^* - L$ be the complement of a language $L$. If $L$ is a regular language, then so is $\overline{L}$. 
Theorem (4.5)

Let \( \overline{L} = \Sigma^* - L \) be the complement of a language \( L \). If \( L \) is a regular language, then so is \( \overline{L} \).

Proof.

Let

\[
A = (Q, \Sigma, \delta, q_0, F)
\]

be a DFA such that \( L(A) = L \). Then

\[
B = (Q, \Sigma, \delta, q_0, Q - F)
\]

is a DFA such that \( L(B) = \overline{L} \).
Question

“Do you see how to take a regular expression and change it into one that defines the complement language?” [Hopcroft, Motwani and Ullman 2007, p. 136]
Closure Under Union and Complementation

Using the closure properties we can prove that a language is not regular.

Example

Given that $L = \{w \in \{0, 1\}^* | w$ has an equal numbers of $0$'s and $1$'s $\}$ is a language not regular. Prove that $L \neq \{w \in \{0, 1\}^* | w$ has an unequal numbers of $0$'s and $1$'s $\}$ is a language not regular.

Proof

Whiteboard.
Using the closure properties we can prove that a language is not regular.

**Example**

Given that

\[ L_\text{=} = \{ w \in \{0, 1\}^* \mid w \text{ has an equal numbers of 0's and 1's} \} \]

is a language not regular. Prove that

\[ L_\text{\neq} = \{ w \in \{0, 1\}^* \mid w \text{ has an unequal numbers of 0's and 1's} \} \]

is a language not regular.
Using the closure properties we can prove that a language is not regular.

Example
Given that

\[ L_\neq = \{ w \in \{0, 1\}^* \mid w \text{ has an equal numbers of 0's and 1's} \} \]

is a language not regular. Prove that

\[ L_{\neq} = \{ w \in \{0, 1\}^* \mid w \text{ has an unequal numbers of 0's and 1's} \} \]

is a language not regular.

Proof
Whiteboard.
Product Construction

Let $A_L$, $A_M$ and $A$ be DFAs given by

\[
A_L = (Q_L, \Sigma, \delta_L, q_L, F_L),
\]
\[
A_M = (Q_M, \Sigma, \delta_M, q_M, F_M),
\]
\[
A = (Q_L \times Q_M, \Sigma, \delta, (q_L, q_M), F_L \times F_M),
\]

where

\[
\delta : (Q_L \times Q_M) \times \Sigma \rightarrow Q_L \times Q_M
\]
\[
\delta((p, q), a) = (\delta_L(p, a), \delta_M(q, a)).
\]
Product Construction

Let $A_L$, $A_M$ and $A$ be DFAs given by

$$A_L = (Q_L, \Sigma, \delta_L, q_L, F_L),$$
$$A_M = (Q_M, \Sigma, \delta_M, q_M, F_M),$$
$$A = (Q_L \times Q_M, \Sigma, \delta, (q_L, q_M), F_L \times F_M),$$

where

$$\delta : (Q_L \times Q_M) \times \Sigma \rightarrow Q_L \times Q_M$$
$$\delta((p, q), a) = (\delta_L(p, a), \delta_M(q, a)).$$

Theorem (Exercise 4.2.15)

For all $w \in \Sigma^*$,

$$\hat{\delta}((q_L, q_M), w) = (\hat{\delta}_L(q_L, w), \hat{\delta}_M(q_M, w)).$$

Closure Properties of Regular Languages
Proof by induction on $w$.  

1. Basis step

$$\hat{\delta}((q_L, q_M), \varepsilon) = (q_L, q_M)$$  

$$= (\delta_L(q_L, \varepsilon), \delta_M(q_M, \varepsilon))$$  

(def. of $\delta$)  

(def. of $\delta_L$ and $\delta_M$)
Product Construction

Proof by induction on $w$.

1. Basis step

$$\hat{\delta}((q_L, q_M), \varepsilon) = (q_L, q_M)$$  
   (def. of $\hat{\delta}$)

$$= (\delta_L(q_L, \varepsilon), \delta_M(q_M, \varepsilon))$$  
   (def. of $\delta_L$ and $\delta_M$)

2. Inductive step

$$\hat{\delta}((q_L, q_M), xa)$$

$$= \delta(\hat{\delta}((q_L, q_M), x), a)$$  
   (def. of $\hat{\delta}$)

$$= \delta((\delta_L(q_L, x), \delta_M(q_M, x)), a)$$  
   (by IH)

$$= (\delta_L(\delta_L(q_L, x), a), \delta_M(\delta_M(q_M, x), a))$$  
   (def. of $\delta$)

$$= (\hat{\delta}_L(q_L, xa), \hat{\delta}_M(q_M, xa))$$  
   (def. of $\hat{\delta}_L$ and $\hat{\delta}_L$)
Theorem (4.8)

If $L$ and $M$ are regular languages, then so is $L \cap M$. 
Closure Under Intersection

Theorem (4.8)
If \( L \) and \( M \) are regular languages, then so is \( L \cap M \).

Proof.
Let \( A_L \) and \( A_M \) be DFAs accepting \( L \) and \( M \). The product construction of \( A_L \) and \( A_M \) accepts \( L \cap M \).
Closure Under Intersection

Theorem (4.8)

If $L$ and $M$ are regular languages, then so is $L \cap M$.

Proof.

Let $A_L$ and $A_M$ be DFAs accepting $L$ and $M$. The product construction of $A_L$ and $A_M$ accepts $L \cap M$.

Different proof.

The regular languages are closure under union and complement, and

$$L \cap M = \overline{L \cup M}.$$
Closure Under Reversal

Definition

Let $w = a_1 a_2 \cdots a_n$ be a word. The reversal of $w$ is defined by

$$w^R = a_n a_{n-1} \cdots a_1.$$
Closure Under Reversal

Definition
Let \( w = a_1 a_2 \cdots a_n \) be a word. The \textbf{reversal} of \( w \) is defined by

\[
w^R = a_n a_{n-1} \cdots a_1.
\]

Definition
Let \( L \) be a language on alphabet \( \Sigma \). The \textbf{reversal} of \( L \) is defined by

\[
L^R = \{ w^R \in \Sigma^* \mid w \in L \}.
\]
Closure Under Reversal

Definition
Let \( w = a_1 a_2 \cdots a_n \) be a word. The reversal of \( w \) is defined by

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 w^R = a_n a_{n-1} \cdots a_1.
\]

Definition
Let \( L \) be a language on alphabet \( \Sigma \). The reversal of \( L \) is defined by

\[
 L^R = \{ w^R \in \Sigma^* \mid w \in L \}.
\]

Theorem (4.11)
If \( L \) is regular language, then so is \( L^R \) (proof using automata or regular expressions)
Closure Under Reversal

Proof using automata.

Let $L$ be recognized by a finite automaton $A$. Turn $A$ into finite automaton for $L^R$, by

1. Reversing all arcs.
2. Make the start state of $A$ be the only accepting state.
3. Create a new start state $p_0$ with transitions $\delta(p_0, \varepsilon) = f$, where $f \in F$ are the accepting states of $A$. 

\[\square\]
Example

A NFA accepting all the binary strings that end in 01.
Closure Under Reversal

Example

A NFA accepting all the binary strings that end in 01.

\[
\begin{align*}
\text{start} & \quad q_0 \quad 0 \quad q_1 \quad 1 \quad q_2 \\
& \quad 0, 1
\end{align*}
\]

A NFA accepting all the binary strings that start with 10.

\[
\begin{align*}
q_0 & \quad 0, 1 \\
& \quad 0 \quad q_1 \quad 1 \quad q_2 \quad \varepsilon \\
p_0 & \quad \text{start}
\end{align*}
\]
Homomorphisms

Definition

An algebraic structure on a set $A \neq \emptyset$ is essentially a collection of $n$-ary operations on $A$ [Birkhoff 1946, 1987].
Homomorphisms

Definition

An algebraic structure on a set $A \neq \emptyset$ is essentially a collection of $n$-ary operations on $A$ [Birkhoff 1946, 1987].

Example (Semigroup)

A semigroup $(S, \ast)$ is a set $S$ with an associative binary operation $\ast : S \times S \rightarrow S$. 

Example (Monoid)

A monoid $(M, \ast, \epsilon)$ is a semigroup $(M, \ast)$ with an element $\epsilon \in M$ which is an unit for $\ast$, i.e. $\forall x. x \ast \epsilon = \epsilon \ast x = x$. 

Closure Properties of Regular Languages
Homomorphisms

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An algebraic structure on a set $A \neq \emptyset$ is essentially a collection of $n$-ary operations on $A$ [Birkhoff 1946, 1987].

Example (Semigroup)
A semigroup $(S, \ast)$ is a set $S$ with an associative binary operation $\ast : S \times S \rightarrow S$.

Example (Monoid)
A monoid $(M, \ast, \varepsilon)$ is a semigroup $(M, \ast)$ with an element $\varepsilon \in M$ which is an unit for $\ast$, i.e. $\forall x. x \ast \varepsilon = \varepsilon \ast x = x$. 
Homomorphisms

Definition

A **homomorphism** is a structure-preserving map between two algebraic structures.
Homomorphisms

Definition

A **homomorphism** is a structure-preserving map between two algebraic structures.

Example

A homomorphism between two semigroups \((S, \ast)\) and \((S', \ast')\) is a function \(\varphi : S \to S'\) such that:

\[
\forall x \ y. \varphi(x \ast y) = \varphi(x) \ast' \varphi(y).
\]

Graphically (see whiteboard).
Homomorphisms

Example

A homomorphism between two monoids \((M, *, \varepsilon)\) and \((M', *, \varepsilon')\) is a function \(\varphi : M \rightarrow M'\) such that:

\[
\forall x \, y. \ \varphi(x * y) = \varphi(x) *' \varphi(y),
\]

\[
\varphi(\varepsilon) = \varepsilon'.
\]
Homomorphisms

Definition

A homomorphism $\varphi$ between two algebraic structures is [Cohn 1981]:

- a **monomorphism** if $\varphi$ is an injection,
- an **epimorphism** if $\varphi$ is a surjection,
- an **endomorphism** if $\varphi$ is from an algebraic structure to itself,
- an **isomorphism** if $\varphi$ is a bijection,
- an **automorphism** if $\varphi$ is a bijective endomorphism.

Closure Under Homomorphism

**Definition**

Let $\Sigma$ and $\Gamma$ be two alphabets. A **homomorphism** between $(\Sigma^*, \cdot, \varepsilon)$ and $(\Gamma^*, \cdot, \varepsilon)$ is a function

$$h : \Sigma^* \rightarrow \Gamma^*$$

$$a_1a_2 \cdots a_n \mapsto h(a_1)h(a_2) \cdots h(a_n)$$

$$\varepsilon \mapsto \varepsilon$$

**Note:** For this reason the textbook talk about a homomorphism $h : \Sigma \rightarrow \Gamma^*$. 
Example
Let \( h : \{0, 1\}^* \rightarrow \{a, b\}^* \) be a homomorphism defined by

\[
\begin{align*}
h(0) &= ab, \\
h(1) &= \varepsilon.
\end{align*}
\]

Then

\[
\begin{align*}
h(0011) &= h(0)h(0)h(1)h(1) \\
&= abab.
\end{align*}
\]
Closure Under Homomorphism

Definition

Let $L$ be a language over an alphabet $\Sigma$ and let $h$ be a homomorphism on $\Sigma$. The **application** of $h$ to $L$, denoted $h(L)$, is defined by†

$$h(L) = \{h(w) \mid w \in L\}.$$ 

†Figure from Hopcroft, Motwani and Ullman [2007, Fig. 4.5a].
Closure Under Homomorphism

Example
Let $h : \{0, 1\}^* \rightarrow \{a, b\}^*$ be a homomorphism defined by

\[
\begin{align*}
  h(0) &= ab, \\
  h(1) &= \varepsilon.
\end{align*}
\]

If $L = L(10^*1)$, then $h(L) = L((ab)^*)$. 
Theorem (4.14)

If $L$ is a regular language over the alphabet $\Sigma$ and $h$ is a homomorphism on $\Sigma$, then $h(L)$ is also regular.
Theorem (4.14)

If $L$ is a regular language over the alphabet $\Sigma$ and $h$ is a homomorphism on $\Sigma$, then $h(L)$ is also regular.

Proof plan

- Let $E$ be a regular expression such that $L = L(E)$.
- Let $h(E)$ be the regular expression replacing each symbol $a \in \Sigma$ by $h(a)$ in the regular expression $E$.
- We need to prove that $L(h(E)) = h(L(E))$. 
Closure Under Homomorphism

Proving \( L(h(E)) = h(L(E)) \).

Basis step

- \( E \) is \( \varepsilon \) or \( \emptyset \).
  1. \( h(E) = E \) (\( h \) does not affect \( E \))
  2. \( h(L(E)) = L(E) \) (\( L(E) \) is empty or only contains \( \varepsilon \))
  3. \( L(h(E)) = L(E) = h(L(E)) \) (by 1 and 2)
Closure Under Homomorphism

Proving $L(h(E)) = h(L(E))$.

Basis step

- $E$ is $\varepsilon$ or $\emptyset$.
  1. $h(E) = E$ ($h$ does not affect $E$)
  2. $h(L(E)) = L(E)$ ($L(E)$ is empty or only contains $\varepsilon$)
  3. $L(h(E)) = L(E) = h(L(E))$ (by 1 and 2)

- $E = a$
  1. $L(E) = \{a\}$
  2. $h(L(E)) = \{h(a)\}$
  3. $h(E)$ is the regular expression that is the string of symbols $h(a)$
  4. $L(h(E)) = \{h(a)\}$
  5. $L(h(E)) = h(L(E))$ (by transitivity between 2 and 4)
Closure Under Homomorphism

Proving $L(h(E)) = h(L(E))$.

Inductive step

1. $L(E) = L(F) \cup L(G)$ (def. of $+$)
2. $h(E) = h(F + G) = h(F) + h(G)$ (def. of $h(E)$)
3. $L(h(E)) = L(h(F) + h(G)) = L(h(F)) \cup L(h(G))$ (def. of $+$)
4. $h(L(E)) = h(L(F) \cup L(G)) = h(L(F)) \cup h(L(G))$ ($h$ is applied to a language by application to each of its strings)
5. $L(h(F)) = h(L(F))$ and $L(h(G)) = h(L(G)$ (IH)
6. $L(h(E)) = h(L(E))$
Closure Under Homomorphism

Proving \( L(h(E)) = h(L(E)) \).

Inductive step (cont.)

1. \( E = FG \) (similar to the previous case)
Closure Under Homomorphism

Proving \( L(h(E)) = h(L(E)) \).

Inductive step (cont.)

- \( E = FG \) (similar to the previous case)
- \( E = F^* \) (similar to the previous case)

1. \( L(E) = (L(F))^* \) (def. of *)
2. \( h(E) = h(F^*) = (h(F))^* \) (def. of \( h(E) \))
3. \( L(h(E)) = L((h(F))^*) = (L(h(F)))^* \) (def. of *)
4. \( h(L(E)) = h((L(F))^*) = (h(L(F)))^* \) (\( h \) is applied to a language by application to each of its strings)
5. \( L(h(F)) = h(L(F)) \) (IH)
6. \( L(h(E)) = h(L(E)) \)
Example

Prove that

\[ L = \{0^i1^j2^k \mid i, j, k \in \mathbb{Z}^+ \text{ and } i \neq j \neq k \} \]

is a language not regular.
Example

Prove that

\[ L = \{0^i1^j2^k \mid i, j, k \in \mathbb{Z}^+ \text{ and } i \neq j \neq k \}\]

is a language not regular.

Proof.

1. We define the homomorphism

\[ h(0) = 0, \quad h(1) = 1, \quad h(2) = \varepsilon. \]
Example
Prove that
\[ L = \{0^i1^j2^k \mid i, j, k \in \mathbb{Z}^+ \text{ and } i \neq j \neq k \} \]
is a language not regular.

Proof.
1. We define the homomorphism
\[ h(0) = 0, \quad h(1) = 1, \quad h(2) = \varepsilon. \]
2. The homomorphism \( h \) removes the \( 2^k \)'s, so
\[ h(L) = \{0^i1^j \mid i, j \in \mathbb{Z}^+ \text{ and } i \neq j\}. \]
Closure Under Homomorphism

Example

Prove that

\[ L = \{0^i1^j2^k \mid i, j, k \in \mathbb{Z}^+ \text{ and } i \neq j \neq k \} \]

is a language not regular.

Proof.

1. We define the homomorphism

\[ h(0) = 0, \quad h(1) = 1, \quad h(2) = \varepsilon. \]

2. The homomorphism \( h \) removes the \( 2^k \)s, so

\[ h(L) = \{0^i1^j \mid i, j \in \mathbb{Z}^+ \text{ and } i \neq j\}. \]

3. We know that \( h(L) \) is not regular, so \( L \) is not regular.
Closure Under Homomorphism

Example
Let $L$ be a regular language and $h$ a homomorphism on $L$. Define $h^*(L)$ by
\[ h^*(L) = L \cup h(L) \cup h(h(L)) \cup h(h(h(L))) \cup ... \]

Is $h^*(L)$ necessarily regular?

†From somewhere in Internet (I don’t remember).
Closure Under Homomorphism

Example

Let $L$ be a regular language and $h$ a homomorphism on $L$. Define $h^*(L)$ by

$$h^*(L) = L \cup h(L) \cup h(h(L)) \cup h(h(h(L))) \cup \ldots$$

Is $h^*(L)$ necessarily regular?

No. Let $L = \{01\}$ and $h$ defined as $h(0) = 00$ and $h(1) = 11$. Then

$$h^*(L) = \{01, 0011, 00001111, \ldots \}$$

$$= \{0^n1^n \mid n = 2^k \text{ for } k \geq 0\},$$

which is a language not regular.†

†From somewhere in Internet (I don’t remember).
Closure Under Inverse Homomorphism

Definition

Let \( h : \Sigma^* \rightarrow \Gamma^* \) be a homomorphism and \( L \subseteq \Gamma^* \) a language. The application of \( h^{-1} \) to \( L \), denoted \( h^{-1}(L) \), is defined by†

\[
h^{-1}(L) = \{ w \in \Sigma^* \mid h(w) \in L \}. \]

Remark

Note that \( h^{-1} \) is a relation but it is not necessarily a function.

†Figure from Hopcroft, Motwani and Ullman [2007, Fig. 4.5b].
Example
Let \( h : \{a, b\} \rightarrow \{0, 1\}^\ast \) a homomorphism defined by

\[
\begin{align*}
h(a) & = 01, \\
h(b) & = 10.
\end{align*}
\]

If \( L = L((00 + 1)^\ast) \), then \( h^{-1}(L) = L((ba)^\ast) \)

Note that \( h^{-1} \) is not a function, but a relation.

(It is necessary to prove \( h(w) \in L \iff w = baba \cdots ba \)).
Theorem (4.16)

Let $h : \Sigma^* \to \Gamma^*$ be a homomorphism and $L \subseteq \Gamma^*$ a regular language. Then $h^{-1}(L)$ is regular (proof using automata).
Example

Prove that $L = \{0^n1^{2n} \mid n \geq 0\}$ is a language not regular.
Example

Prove that \( L = \{0^n1^{2n} \mid n \geq 0\} \) is a language not regular.

Proof.

1. Given the homomorphism

   \[
   h(0) = 0, \\
   h(1) = 11,
   \]

   then \( h^{-1}(L) = \{0^n1^n \mid n \geq 0\} \).
Example

Prove that $L = \{0^n1^{2n} \mid n \geq 0\}$ is a language not regular.

Proof.

1. Given the homomorphism

   \[ h(0) = 0, \]
   \[ h(1) = 11, \]

   then $h^{-1}(L) = \{0^n1^n \mid n \geq 0\}$.  

2. Since $h^{-1}(L)$ is not regular, then $L$ is not regular.
Exercise (4.2.2)

If $L$ is a language, and $a$ is a symbol, then $L/a$, the quotient of $L$ and $a$, is the set of strings $w$ such that $wa$ is in $L$. For example, if $L = \{a, aab, baa\}$, then $L/a = \{\varepsilon, ba\}$. Prove that if $L$ is regular, so is $L/a$. **Hint:** Start with a DFA for $L$ and consider the set of accepting states.
Exercise (4.2.2)

If $L$ is a language, and $a$ is a symbol, then $L/a$, the quotient of $L$ and $a$, is the set of strings $w$ such that $wa$ is in $L$. For example, if $L = \{a, aab, baa\}$, then $L/a = \{\varepsilon, ba\}$. Prove that if $L$ is regular, so is $L/a$. Hint: Start with a DFA for $L$ and consider the set of accepting states.

Proof (Hopcroft, Motwani and Ullman [2007] solution).

Start with a DFA $A$ for $L$. Construct a new DFA $B$, that is exactly the same as $A$, except that state $q$ is an accepting state of $B$ if and only if $\delta(q, a)$ is an accepting state of $A$. Then $B$ accepts input string $w$ if and only if $A$ accepts $wa$; that is, $L(B) = L/a$. $\blacksquare$
Exercise (4.2.3)

If $L$ is a language, and $a$ is a symbol, then $a \backslash L$ is the set of strings $w$ such that $aw$ is in $L$. For example, if $L = \{a, aab, baa\}$, then $a \backslash L = \{\varepsilon, ab\}$. Prove that if $L$ is regular, so is $a \backslash L$. Hint: Start with a DFA for $L$ and consider its start state.
Closure Properties

Exercise (4.2.3)

If $L$ is a language, and $a$ is a symbol, then $a\setminus L$ is the set of strings $w$ such that $aw$ is in $L$. For example, if $L = \{a, aab, baa\}$, then $a\setminus L = \{\varepsilon, ab\}$. Prove that if $L$ is regular, so is $a\setminus L$. *Hint*: Start with a DFA for $L$ and consider its start state.

Proof (Hopcroft, Motwani and Ullman [2007] solution).

Start with a DFA $A$ for $L$. Construct a new DFA $B$, that is exactly the same as $A$, except that its start state is $\delta(q_0, a)$ where $q_0$ is the start state of $A$. Then $B$ accepts input string $w$ if and only if $A$ accepts $aw$; that is, $L(B) = L\setminus a$. 

$\blacksquare$
References


