Let $L$ and $M$ be regular languages. The following languages are regular:
Introduction

Let $L$ and $M$ be regular languages. The following languages are regular:

- Union: $L \cup M$
Introduction

Let $L$ and $M$ be regular languages. The following languages are regular:

- Union: $L \cup M$
- Intersection: $L \cap M$
Introduction

Let $L$ and $M$ be regular languages. The following languages are regular:

- **Union**: $L \cup M$
- **Intersection**: $L \cap M$
- **Complement**: $\overline{L}$
Let $L$ and $M$ be regular languages. The following languages are regular:

- **Union**: $L \cup M$
- **Intersection**: $L \cap M$
- **Complement**: $\overline{L}$
- **Difference**: $L - M$
Let $L$ and $M$ be regular languages. The following languages are regular:

- **Union**: $L \cup M$
- **Intersection**: $L \cap M$
- **Complement**: $\overline{L}$
- **Difference**: $L - M$
- **Reversal**: $L^R = \{w^r \in \Sigma^* \mid w \in L\}$

**Introduction**
Introduction

Let $L$ and $M$ be regular languages. The following languages are regular:

- **Union:** $L \cup M$
- **Intersection:** $L \cap M$
- **Complement:** $\overline{L}$
- **Difference:** $L - M$
- **Reversal:** $L^R = \{w^r \in \Sigma^* \mid w \in L\}$
- **Closure:** $L^*$
Let $L$ and $M$ be regular languages. The following languages are regular:

- **Union**: $L \cup M$
- **Intersection**: $L \cap M$
- **Complement**: $\overline{L}$
- **Difference**: $L - M$
- **Reversal**: $L^R = \{ w^r \in \Sigma^* \mid w \in L \}$
- **Closure**: $L^*$
- **Concatenation**: $LM$
Introduction

Let $L$ and $M$ be regular languages. The following languages are regular:

- **Union**: $L \cup M$
- **Intersection**: $L \cap M$
- **Complement**: $\overline{L}$
- **Difference**: $L - M$
- **Reversal**: $L^R = \{w^r \in \Sigma^* \mid w \in L\}$
- **Closure**: $L^*$
- **Concatenation**: $LM$
- **Homomorphism**: $h(L) = \{h(w) \mid w \in L \text{ and } h \text{ is a homomorphism}\}$
Let $L$ and $M$ be regular languages. The following languages are regular:

- Union: $L \cup M$
- Intersection: $L \cap M$
- Complement: $\overline{L}$
- Difference: $L - M$
- Reversal: $L^R = \{w^r \in \Sigma^* \mid w \in L\}$
- Closure: $L^*$
- Concatenation: $LM$
- Homomorphism: $h(L) = \{h(w) \mid w \in L \text{ and } h \text{ is a homomorphism}\}$
- Inverse homomorphism: $h^{-1}(L) = \{w \in \Sigma \mid h(w) \in L \text{ and } h \text{ is a homomorphism}\}$
Closure Under Union and Complementation

Theorem (Hopcroft, Motwani and Ullman [2007], Theorem 4.4)
If $L$ and $M$ are regular languages, then so is $L \cup M$. 
Closure Under Union and Complementation

Theorem (Hopcroft, Motwani and Ullman [2007], Theorem 4.4)

If \( L \) and \( M \) are regular languages, then so is \( L \cup M \).

Proof

(Using regular expressions)
Closure Under Union and Complementation

Theorem (Hopcroft, Motwani and Ullman [2007], Theorem 4.5)

Let $\overline{L} = \Sigma^* - L$ be the complement of a language $L$. If $L$ is a regular language, then so is $\overline{L}$. 

Proof.

Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA such that $L(A) = L$. Then $B = (Q, \Sigma, \delta, q_0, Q - F)$ is a DFA such $L(B) = \overline{L}$. 


Closure Under Union and Complementation

Theorem (Hopcroft, Motwani and Ullman [2007], Theorem 4.5)

Let $\overline{L} = \Sigma^* - L$ be the complement of a language $L$. If $L$ is a regular language, then so is $\overline{L}$.

Proof.

Let

$$A = (Q, \Sigma, \delta, q_0, F)$$

be a DFA such that $L(A) = L$. Then

$$B = (Q, \Sigma, \delta, q_0, Q - F)$$

is a DFA such $L(B) = \overline{L}$. 

\[ ~\]
Closure Under Union and Complementation

Theorem (Hopcroft, Motwani and Ullman [2007], Theorem 4.5)

Let $\overline{L} = \Sigma^* - L$ be the complement of a language $L$. If $L$ is a regular language, then so is $\overline{L}$.

Question: “Do you see how to take a regular expression and change it into one that defines the complement language?”*

Closure Under Union and Complementation

Using the closure properties we can prove that a language is not regular.

Example

Given that $L = \{ w \in \{0, 1\}^* \mid w$ has an equal numbers of $0$'s and $1$'s $\}$ is a language not regular. Prove that $L' = \{ w \in \{0, 1\}^* \mid w$ has an unequal numbers of $0$'s and $1$'s $\}$ is a language not regular.

Proof

Whiteboard.
Closure Under Union and Complementation

Using the closure properties we can prove that a language is not regular.

Example
Given that

\[ L_\subseteq = \{ w \in \{0, 1\}^* \mid w \text{ has an equal numbers of 0's and 1's} \} \]

is a language not regular. Prove that

\[ L_\neq = \{ w \in \{0, 1\}^* \mid w \text{ has an unequal numbers of 0's and 1's} \} \]

is a language not regular.
Closure Under Union and Complementation

Using the closure properties we can prove that a language is not regular.

Example

Given that

\[ L_\equiv = \{ w \in \{0, 1\}^* \mid w \text{ has an equal numbers of 0's and 1's} \} \]

is a language not regular. Prove that

\[ L_\neq = \{ w \in \{0, 1\}^* \mid w \text{ has an unequal numbers of 0's and 1's} \} \]

is a language not regular.

Proof

Whiteboard.
Product Construction

Let $A_L$, $A_M$ and $A$ be DFAs given by

$$A_L = (Q_L, \Sigma, \delta_L, q_L, F_L),$$
$$A_M = (Q_M, \Sigma, \delta_M, q_M, F_M),$$
$$A = (Q_L \times Q_M, \Sigma, \delta, (q_L, q_M), F_L \times F_M),$$

where $\delta((p, q), a) = (\delta_L(p, a), \delta_M(q, a))$. 

Theorem (Hopcroft, Motwani and Ullman [2007], Exercise 4.2.15)

For all $w \in \Sigma^*$,

$$\hat{\delta}((q_L, q_M), w) = (\hat{\delta}_L(q_L, w), \hat{\delta}_M(q_M, w)).$$
Product Construction

Let $A_L$, $A_M$ and $A$ be DFAs given by

$$A_L = (Q_L, \Sigma, \delta_L, q_L, F_L),$$
$$A_M = (Q_M, \Sigma, \delta_M, q_M, F_M),$$
$$A = (Q_L \times Q_M, \Sigma, \delta, (q_L, q_M), F_L \times F_M),$$

where $\delta((p, q), a) = (\delta_L(p, a), \delta_M(q, a))$.

Theorem (Hopcroft, Motwani and Ullman [2007], Exercise 4.2.15)
For all $w \in \Sigma^*$,

$$\hat{\delta}((q_L, q_M), w) = (\hat{\delta}_L(q_L, w), \hat{\delta}_M(q_M, w)).$$
Product Construction

Proof by induction on $w$.

1. Basis step

   \[
   \hat{\delta}((q_L, q_M), \varepsilon) = (q_L, q_M) \quad \text{(def. of } \hat{\delta})
   \]

   \[
   = (\delta_L(q_L, \varepsilon), \delta_M(q_M, \varepsilon)) \quad \text{(def. of } \delta_L \text{ and } \delta_M)
   \]
Proof by induction on $w$.

1. Basis step

$$\hat{\delta}((q_L, q_M), \varepsilon) = (q_L, q_M)$$  
   
   $(\text{def. of } \hat{\delta})$

   $$= (\hat{\delta}_L(q_L, \varepsilon), \hat{\delta}_M(q_M, \varepsilon))$$  
   
   $(\text{def. of } \hat{\delta}_L \text{ and } \hat{\delta}_M)$

2. Inductive step

$$\hat{\delta}((q_L, q_M), xa)$$

$$= \delta(\hat{\delta}((q_L, q_M), x), a)$$  
   
   $(\text{def. of } \hat{\delta})$

   $$= \delta((\hat{\delta}_L(q_L, x), \hat{\delta}_M(q_M, x)), a)$$  
   
   $(\text{by IH})$

   $$= (\delta_L(\hat{\delta}_L(q_L, x), a), \delta_M(\hat{\delta}_M(q_M, x), a))$$  
   
   $(\text{def. of } \delta)$

   $$= (\hat{\delta}_L(q_L, xa), \hat{\delta}_M(q_M, xa))$$  
   
   $(\text{def. of } \hat{\delta}_L \text{ and } \hat{\delta}_L)$
Closure Under Intersection

Theorem (Hopcroft, Motwani and Ullman [2007], Theorem 4.8)

If $L$ and $M$ are regular languages, then so is $L \cap M$. 

Proof. Let $A_L$ be a DFA accepting $L$ and $A_M$ be a DFA accepting $M$. The product construction of $A_L$ and $A_M$ accepts $L \cap M$. 

Different proof. The regular languages are closure under union and complement, and $L \cap M = L \cup M$. 

Closure Under Intersection

Theorem (Hopcroft, Motwani and Ullman [2007], Theorem 4.8)
If $L$ and $M$ are regular languages, then so is $L \cap M$.

Proof.
Let $A_L$ and $A_M$ be DFAs accepting $L$ and $M$. The product construction of $A_L$ and $A_M$ accepts $L \cap M$. 

[End of proof]
Closure Under Intersection

Theorem (Hopcroft, Motwani and Ullman [2007], Theorem 4.8)
If $L$ and $M$ are regular languages, then so is $L \cap M$.

Proof.
Let $A_L$ and $A_M$ be DFAs accepting $L$ and $M$. The product construction of $A_L$ and $A_M$ accepts $L \cap M$.

Different proof.
The regular languages are closure under union and complement, and

\[ L \cap M = \overline{L} \cup \overline{M}. \]
Closure Under Reversal

Definition (Reversal of a language)

If \( w = a_1 a_2 \cdots a_n \) then the reversal of \( w \) is defined by \( w^R = a_n a_{n-1} \cdots a_1 \).

\( L^R = \{ w^R \in \Sigma^* \mid w \in L \} \) is the reversal of the language \( L \) on the alphabet \( \Sigma \).
Closure Under Reversal

Definition (Reversal of a language)

If $w = a_1a_2 \cdots a_n$ then the reversal of $w$ is defined by $w^R = a_na_{n-1} \cdots a_1$.

$L^R = \{w^R \in \Sigma^* \mid w \in L\}$ is the reversal of the language $L$ on the alphabet $\Sigma$.

Theorem (Hopcroft, Motwani and Ullman [2007], Theorem 4.11)

If $L$ is regular language, then so is $L^R$ (proof using automata or regular expressions).
Closure Under Reversal

Proof using automata.

Let $L$ be recognized by a finite automaton $A$. Turn $A$ into finite automaton for $L^R$, by

1. Reversing all arcs.
2. Make the start state of $A$ be the only accepting state.
3. Create a new start state $p_0$ with transitions $\delta(p_0, \varepsilon) = f$, where $f \in F$ are the accepting states of $A$. 

$\blacksquare$
Example

A NFA accepting all the binary strings that end in 01.
Closure Under Reversal

Example

A NFA accepting all the binary strings that end in 01.

A NFA accepting all the binary strings that start with 10.
Homomorphisms

Definition (Algebraic structure)

An algebraic structure on a set $A \neq \emptyset$ is essentially a collection of $n$-ary operations on $A$ [Cohn 1981, p. 41].
Homomorphisms

Definition (Algebraic structure)

An algebraic structure on a set $A \neq \emptyset$ is essentially a collection of $n$-ary operations on $A$ [Cohn 1981, p. 41].

Example (Semigroup)

A semigroup $(S, \ast)$ is a set $S$ with an associative binary operation $\ast : S \times S \rightarrow S$. 

Example (Monoid)

A monoid $(M, \ast, \varepsilon)$ is a semigroup $(M, \ast)$ with an element $\varepsilon \in M$ which is an unit for $\ast$, i.e. $\forall x. x \ast \varepsilon = \varepsilon \ast x = x$. 
Homomorphisms

Definition (Algebraic structure)

An algebraic structure on a set \( A \neq \emptyset \) is essentially a collection of \( n \)-ary operations on \( A \) [Cohn 1981, p. 41].

Example (Semigroup)

A semigroup \((S, \ast)\) is a set \( S \) with an associative binary operation \( \ast : S \times S \to S \).

Example (Monoid)

A monoid \((M, \ast, \varepsilon)\) is a semigroup \((M, \ast)\) with an element \( \varepsilon \in M \) which is an unit for \( \ast \), i.e. \( \forall x. \ x \ast \varepsilon = \varepsilon \ast x = x \).
Homomorphisms

Definition (homomorphism)

A homomorphism is a structure-preserving map between two algebraic structures.
Homomorphisms

Definition (homomorphism)

A *homomorphism* is a structure-preserving map between two algebraic structures.

Example

A homomorphism between two semigroups \((S, \ast)\) and \((S', \ast')\) is a function \(f : S \rightarrow S'\) such that:

\[
\forall x \ y.\ f(x \ast y) = f(x) \ast' f(x).
\]

Graphically (see whiteboard).
Homomorphisms

Example

A homomorphism between two monoids \((M, \ast, \varepsilon)\) and \((M', \ast', \varepsilon')\) is a function \(f : M \to M'\) such that:

\[
\forall x \ y. \ f(x \ast y) = f(x) \ast' f(y),
\]

\[
f(\varepsilon) = \varepsilon'.
\]
Homomorphisms

A homomorphism between two algebraic structures is a:

- **Monomorphism** if $f$ is an injection
- **Epimorphism** if $f$ is an surjection
- **Isomorphism** if $f$ is bijection
- **Endomorphism** if the homomorphism is from an algebraic structure to itself
- **Automorphism** if the homomorphism is a bijective endomorphism
Closure Under Homomorphism

Definition

Let $\Sigma$ and $\Gamma$ be two alphabets. A homomorphism between $(\Sigma^*, \cdot, \varepsilon)$ and $(\Gamma^*, \cdot, \varepsilon)$ is a function

$$h : \Sigma^* \rightarrow \Gamma^*$$

$$a_1a_2\cdots a_n \mapsto h(a_1)h(a_2)\cdots h(a_n)$$

$$\varepsilon \mapsto \varepsilon$$

Note: For this reason the textbook talk about a homomorphism $h : \Sigma \rightarrow \Gamma^*$. 
Closure Under Homomorphism

Example

Let \( h : \{0, 1\}^* \rightarrow \{a, b\}^* \) be a homomorphism defined by

\[
    h(0) = ab, \\
    h(1) = \varepsilon.
\]

Then

\[
    h(0011) = h(0)h(0)h(1)h(1) = abab.
\]
Closure Under Homomorphism

Definition (Application of a homomorphism to a language)

If $L$ is a language over the alphabet $\Sigma$ and $h$ is a homomorphism on $\Sigma$, then

$$h(L) = \{h(w) \mid w \in L\}.$$ 

*Hopcroft, Motwani and Ullman [2007, Fig. 4.5a].*
Closure Under Homomorphism

Example

Let \( h : \{0, 1\}^* \to \{a, b\}^* \) be a homomorphism defined by

\[
\begin{align*}
    h(0) &= ab, \\
    h(1) &= \varepsilon.
\end{align*}
\]

If \( L = L(10^*1) \), then \( h(L) = L((ab)^*) \).
Closure Under Homomorphism

**Theorem (Hopcroft, Motwani and Ullman [2007], Theorem 4.14)**

If $L$ is a regular language over the alphabet $\Sigma$ and $h$ is a homomorphism on $\Sigma$, then $h(L)$ is also regular.

**Proof plan**

Let $E$ be a regular expression such $L = L(E)$. Let $h(E)$ be the regular expression replacing each symbol $a \in \Sigma$ by $h(a)$ in the regular expression $E$. We need to prove that $L(h(E)) = h(L(E))$. 
Closure Under Homomorphism

Theorem (Hopcroft, Motwani and Ullman [2007], Theorem 4.14)

If $L$ is a regular language over the alphabet $\Sigma$ and $h$ is a homomorphism on $\Sigma$, then $h(L)$ is also regular.

Proof plan

- Let $E$ be a regular expression such $L = L(E)$.
- Let $h(E)$ be the regular expression replacing each symbol $a \in \Sigma$ by $h(a)$ in the regular expression $E$.
- We need to prove that $L(h(E)) = h(L(E))$. 
Closure Under Homomorphism

Proving $L(h(E)) = h(L(E))$.

Basis step

- $E$ is $\varepsilon$ or $\emptyset$.
  1. $h(E) = E$ (h does not affect $E$)
  2. $h(L(E)) = L(E)$ ($L(E)$ is empty or only contains $\varepsilon$)
  3. $L(h(E)) = L(E) = h(L(E))$ (by 1 and 2)
Closure Under Homomorphism

Proving $L(h(E)) = h(L(E))$.

Basis step

- $E$ is $\varepsilon$ or $\emptyset$.
  1. $h(E) = E$ ($h$ does not affect $E$)
  2. $h(L(E)) = L(E)$ ($L(E)$ is empty or only contains $\varepsilon$)
  3. $L(h(E)) = L(E) = h(L(E))$ (by 1 and 2)

- $E = a$
  1. $L(E) = \{a\}$
  2. $h(L(E)) = \{h(a)\}$
  3. $h(E)$ is the regular expression that is the string of symbols $h(a)$
  4. $L(h(E)) = \{h(a)\}$
  5. $L(h(E)) = h(L(E))$ (by transitivity between 2 and 4)
Closure Under Homomorphism

Proving \( L(h(E)) = h(L(E)) \).

Inductive step

1. \( E = F + G \)

2. \( L(E) = L(F) \cup L(G) \) (def. of +)

3. \( h(E) = h(F + G) = h(F) + h(G) \) (def. of \( h(E) \))

4. \( L(h(E)) = L(h(F) + h(G)) = L(h(F)) \cup L(h(G)) \) (def. of +)

5. \( h(L(E)) = h(L(F) \cup L(G)) = h(L(F)) \cup h(L(G)) \) (\( h \) is applied to a language by application to each of its strings)

6. \( L(h(E)) = h(L(E)) \)
Closure Under Homomorphism

Proving $L(h(E)) = h(L(E))$.

Inductive step (cont.)

- $E = FG$ (similar to the previous case)
Closure Under Homomorphism

Proving $L(h(E)) = h(L(E))$.

Inductive step (cont.)

- $E = FG$ (similar to the previous case)
- $E = F^*$ (similar to the previous case)

1. $L(E) = (L(F))^*$ (def. of $*$)
2. $h(E) = h(F^*) = (h(F))^*$ (def. of $h(E)$)
3. $L(h(E)) = L((h(F))^*) = (L(h(F)))^*$ (def. of $*$)
4. $h(L(E)) = h((L(F))^*) = (h(L(F)))^*$ ($h$ is applied to a language by application to each of its strings)
5. $L(h(F)) = h(L(F))$ (IH)
6. $L(h(E)) = h(L(E))$
Closure Under Homomorphism

Example

Prove that

\[ L = \{0^i1^j2^k \mid i, j, k \in \mathbb{Z}^+ \text{ and } i \neq j \neq k \} \]

is a language not regular.
Example

Prove that

\[ L = \{ 0^i 1^j 2^k \mid i, j, k \in \mathbb{Z}^+ \text{ and } i \neq j \neq k \} \]

is a language not regular.

Proof.

1. We define the homomorphism

\[ h(0) = 0, \quad h(1) = 1, \quad h(2) = \varepsilon. \]
Example

Prove that

\[ L = \{0^i1^j2^k \mid i, j, k \in \mathbb{Z}^+ \text{ and } i \neq j \neq k \} \]

is a language not regular.

Proof.

1. We define the homomorphism

\[ h(0) = 0, \quad h(1) = 1, \quad h(2) = \varepsilon. \]

2. The homomorphism \( h \) removes the \( 2^k \)s, so

\[ h(L) = \{0^i1^j \mid i, j \in \mathbb{Z}^+ \text{ and } i \neq j \}. \]
Closure Under Homomorphism

Example

Prove that
\[ L = \{0^i 1^j 2^k \mid i, j, k \in \mathbb{Z}^+ \text{ and } i \neq j \neq k \} \]

is a language not regular.

Proof.

1. We define the homomorphism
\[ h(0) = 0, \quad h(1) = 1, \quad h(2) = \varepsilon. \]

2. The homomorphism \( h \) removes the \( 2^k \)'s, so
\[ h(L) = \{0^i 1^j \mid i, j \in \mathbb{Z}^+ \text{ and } i \neq j\}. \]

3. We know that \( h(L) \) is not regular, so \( L \) is not regular.
Closure Under Homomorphism

Example

Let $L$ be a regular language and $h$ a homomorphism on $L$. Define $h^*(L)$ by

$$h^*(L) = L \cup h(L) \cup h(h(L)) \cup h(h(h(L))) \cup ...$$

Is $h^*(L)$ necessarily regular?

*From somewhere in Internet (I don’t remember).*
Closure Under Homomorphism

Example

Let \( L \) be a regular language and \( h \) a homomorphism on \( L \). Define \( h^*(L) \) by

\[
h^*(L) = L \cup h(L) \cup h(h(L)) \cup h(h(h(L))) \cup \ldots
\]

Is \( h^*(L) \) necessarily regular?

No. Let \( L = \{01\} \) and \( h \) defined as \( h(0) = 00 \) and \( h(1) = 11 \). Then

\[
h^*(L) = \{01, 0011, 00001111, \ldots \}
\]

\[
= \{0^n1^n \mid n = 2^k \text{ for } k \geq 0\},
\]

which is a language not regular.*

*From somewhere in Internet (I don’t remember).
Closure Under Inverse Homomorphism

Definition
Let \( h : \Sigma^* \rightarrow \Gamma^* \) be a homomorphism and \( L \subseteq \Gamma^* \) a language, then:

\[
h^{-1}(L) = \{ w \in \Sigma^* \mid h(w) \in L \}.
\]

*Hopcroft, Motwani and Ullman [2007, Fig. 4.5b].
Closure Under Inverse Homomorphism

Example
Let $h : \{a, b\} \rightarrow \{0, 1\}^*$ a homomorphism defined by

$h(a) = 01,$

$h(b) = 10.$
Closure Under Inverse Homomorphism

Example
Let \( h : \{a, b\} \to \{0, 1\}^* \) a homomorphism defined by

\[
\begin{align*}
h(a) &= 01, \\
h(b) &= 10.
\end{align*}
\]

If \( L = L((00 + 1)^*) \), then \( h^{-1}(L) = L((ba)^*) \)

(It is necessary to prove \( h(w) \in L \iff w = babab\cdots ba \)).
Closure Under Inverse Homomorphism

Theorem (Hopcroft, Motwani and Ullman [2007], Theorem 4.16)

Let $h : \Sigma^* \rightarrow \Gamma^*$ be a homomorphism and $L \subseteq \Gamma^*$ a regular language. Then $h^{-1}(L)$ is regular (proof using automata).
Example

Prove that \( L = \{0^n1^{2n} \mid n \geq 0\} \) is a language not regular.
Example

Prove that \( L = \{0^n1^{2n} \mid n \geq 0\} \) is a language not regular.

Proof.

1. Given the homomorphism

\[
\begin{align*}
    h(0) &= 0, \\
    h(1) &= 11,
\end{align*}
\]

then \( h^{-1}(L) = \{0^n1^n \mid n \geq 0\} \).
Example

Prove that \( L = \{0^n1^{2n} \mid n \geq 0\} \) is a language not regular.

Proof.

1. Given the homomorphism

\[
\begin{align*}
h(0) &= 0, \\
h(1) &= 11,
\end{align*}
\]

then \( h^{-1}(L) = \{0^n1^n \mid n \geq 0\} \).

2. Since \( h^{-1}(L) \) is not regular, then \( L \) is not regular.
Closure Properties

Exercise (Hopcroft, Motwani and Ullman [2007], Exercise 4.2.2)

If \( L \) is a language, and \( a \) is a symbol, then \( L/\alpha \), the quotient of \( L \) and \( a \), is the set of strings \( w \) such that \( wa \) is in \( L \). For example, if \( L = \{a, aab, baa\} \), then \( L/\alpha = \{\epsilon, ba\} \). Prove that if \( L \) is regular, so is \( L/\alpha \). Hint: Start with a DFA for \( L \) and consider the set of accepting states.
Closure Properties

Exercise (Hopcroft, Motwani and Ullman [2007], Exercise 4.2.2)

If $L$ is a language, and $a$ is a symbol, then $L/a$, the quotient of $L$ and $a$, is the set of strings $w$ such that $wa$ is in $L$. For example, if $L = \{a, aab, baa\}$, then $L/a = \{\varepsilon, ba\}$. Prove that if $L$ is regular, so is $L/a$. Hint: Start with a DFA for $L$ and consider the set of accepting states.

Proof (Hopcroft, Motwani and Ullman [2007] solution).

Start with a DFA $A$ for $L$. Construct a new DFA $B$, that is exactly the same as $A$, except that state $q$ is an accepting state of $B$ if and only if $\delta(q, a)$ is an accepting state of $A$. Then $B$ accepts input string $w$ if and only if $A$ accepts $wa$; that is, $L(B) = L/a$. \qed
Closure Properties

Exercise (Hopcroft, Motwani and Ullman [2007], Exercise 4.2.3)

If $L$ is a language, and $a$ is a symbol, then $a\setminus L$ is the set of strings $w$ such that $aw$ is in $L$. For example, if $L = \{a, aab, baa\}$, then $a\setminus L = \{\varepsilon, ab\}$. Prove that if $L$ is regular, so is $a\setminus L$. Hint: Start with a DFA for $L$ and consider its start state.
Exercise (Hopcroft, Motwani and Ullman [2007], Exercise 4.2.3)

If $L$ is a language, and $a$ is a symbol, then $a\setminus L$ is the set of strings $w$ such that $aw$ is in $L$. For example, if $L = \{a, aab, baa\}$, then $a\setminus L = \{\varepsilon, ab\}$. Prove that if $L$ is regular, so is $a\setminus L$. Hint: Start with a DFA for $L$ and consider its start state.

Proof (Hopcroft, Motwani and Ullman [2007] solution).

Start with a DFA $A$ for $L$. Construct a new DFA $B$, that is exactly the same as $A$, except that its start state is $\delta(q_0, a)$ where $q_0$ is the start state of $A$. Then $B$ accepts input string $w$ if and only if $A$ accepts $aw$; that is, $L(B) = L\setminus a$. 

\[ \]