Definition (Equivalence of regular expressions)

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Algebraic Laws for Regular Expressions

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Sugar syntax

\[
L^+ \overset{\text{def}}{=} LL^*, \quad L? \overset{\text{def}}{=} \varepsilon + L.
\]
Some laws for union

\[(L + M) + N = L + (M + N)\]  \hspace{1cm} \text{(associativity)}
\[L + \emptyset = \emptyset + L = L\]  \hspace{1cm} \text{(identity)}
\[L + M = M + L\]  \hspace{1cm} \text{(commutativity)}
\[L + L = L\]  \hspace{1cm} \text{(idempotence)}

Remark: There is no inverse for union.
Algebraic Laws for Regular Expressions

Some laws for concatenation

\[(LM)N = L(MN)\]  \hspace{2cm} \text{(associativity)}

\[L\varepsilon = \varepsilon L = L\]  \hspace{2cm} \text{(identity)}

\[LM \neq ML\]  \hspace{2cm} \text{(non-commutativity)}

\[L\emptyset = \emptyset L = \emptyset\]  \hspace{2cm} \text{(\emptyset is the annihilator for concatenation)}

Remark: There is no inverse for concatenation.
Algebraic Laws for Regular Expressions

Some laws for union and concatenation

\[ L(M + N) = LM + LN \]  \hspace{1cm} \text{(distributive)}

\[ (L + M)N = LN + LM \]  \hspace{1cm} \text{(distributive)}
Algebraic Laws for Regular Expressions

Some laws for closure

\[(L^*)^* = L^*\]  (idempotence)
\[\emptyset^* = \varepsilon\]
\[\varepsilon^* = \varepsilon\]
\[(\varepsilon + L)^* = L^*\]
\[L^* = L^+ + \varepsilon\]
Simplification of Regular Expressions

Example

\[0 + (\varepsilon + 1)(\varepsilon + 1)^*0 = 0 + (\varepsilon + 1)1^*0 \quad ((\varepsilon + L)^* = L^*)\]

\[= 0 + (\varepsilon 1^* + 11^*)0 \quad \text{(distributive)}\]

\[= 0 + (1^* + 11^*)0 \quad \text{(identity)}\]

\[= 0 + (1^* + 1^+)0 \quad \text{(def. } L^+)\]

\[= 0 + 1^*0 \quad \text{(equivalence)}\]

\[= 1^*0 \quad \text{(equivalence)}\]
Discovering Laws for Regular Expressions

Let $E$ and $F$ be two regular expressions with the same set of variables \{\(L_1, \ldots, L_n\)\}.

To test if $E = F$:

1. Convert $E$ and $F$ to concrete regular expressions $C$ and $D$, replacing each $L_i$ by a different symbol $a_i$, for $i = 1, 2, \ldots, n$.

2. Test whether $L(C) = L(D)$. If so, then $E = F$, and if not $E \neq F$. 

Observation

We are proving by example!
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1. We replace the variable $L$ by the concrete regular expression $a$.
2. $a^* \not\equiv a^* a^*$. 
Discovering Laws for Regular Expressions

Example

Prove or disprove that $L^* = L^* L^*$.

1. We replace the variable $L$ by the concrete regular expression $a$.
2. $a^* \equiv a^* a^*$.
3. Because $L(a^*) = L(a^* a^*)$, we conclude that $L^* = L^* L^*$. 
Discovering Laws for Regular Expressions

Example
Prove of disprove that $L + ML = (L + M)L$.
Discovering Laws for Regular Expressions

Example

Prove or disprove that $L + ML = (L + M)L$.

1. We replace the variables $L$ and $M$ by the concrete regular expressions $a$ and $b$ respectively.
Discovering Laws for Regular Expressions

Example

Prove or disprove that \( L + ML = (L + M)L \).

1. We replace the variables \( L \) and \( M \) by the concrete regular expressions \( a \) and \( b \) respectively.

2. \( a + ba \not\equiv (a + b)a \).
Discovering Laws for Regular Expressions

Example

Prove or disprove that $L + ML = (L + M)L$.

1. We replace the variables $L$ and $M$ by the concrete regular expressions $a$ and $b$ respectively.

2. $a + ba \not\equiv (a + b)a$.

3. $aa \notin L(a + ba)$ and $aa \in L((a + b)a)$

   $\Rightarrow L(a + ba) \neq L((a + b)a)$

   $\Rightarrow L + ML \neq (L + M)L$
Discovering Laws for Regular Expressions

Example (Hopcroft, Motwani and Ullman [2007], Exercise 3.4.2.d)

Prove or disprove that \((L + M)^* M \neq (L^* M)^*\).
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Prove or disprove that \((L + M)^* M = (L^* M)^*\).

1. We replace the variables \(L\) and \(M\) by the concrete regular expressions \(a\) and \(b\) respectively.

2. \((a + b)^* b \not\equiv (a^* b)^*\).

3. Since \(\varepsilon \notin (a + b)^* b\) and \(\varepsilon \in (a^* b)^*\)

\[\Rightarrow (L + M)^* M \neq (L^* M)^*\]
Discovering Laws for Regular Expressions

Example (counter-example)

Extensions of the previous test beyond regular expressions may fail.

1. Add $\cap$ to the regular expression operators.
2. Test if $L \cap M \cap N = L \cap M$.
3. From $L = a$, $M = b$ and $N = c$, we should conclude that $L \cap M \cap N = L \cap M$, that is, the "property" is true.
4. The "property" is false. For example, if $L = M = a$ and $N = \emptyset$ then $L \cap M \cap N \neq L \cap M$.
5. Therefore, the test is not valid!
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Derivative of a Regular Expression

Definition (Derivative of a language by a symbol)

Let $L \subseteq \Sigma^*$ be a language and $a \in \Sigma$ a symbol. We define $a\backslash L$ (derivative of $L$ by $a$) by

$$a\backslash L = \{x \in \Sigma^* \mid ax \in L\}.$$
Derivative of a Regular Expression

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Examples

$$a \backslash \{ abab, abba \} = \{ bab, bba \},$$

$$a \backslash L(ab^*) = L(b^*),$$

$$b \backslash L(ab^*) = \emptyset.$$
Derivative of a Regular Expression

Definition (Derivative of a regular expression by a symbol)

We define recursively $a\backslash E$ (derivative of the regular expression $E$ by $a \in \Sigma$) by

$$
\begin{align*}
  a\backslash \emptyset &= \emptyset, \\
  a\backslash \varepsilon &= \emptyset, \\
  a\backslash a &= \varepsilon, \\
  a\backslash b &= \emptyset \text{ for } a \neq b,
\end{align*}
$$
Derivative of a Regular Expression

Definition (Derivative of a regular expression by a symbol)

We define recursively $a \setminus E$ (derivative of the regular expression $E$ by $a \in \Sigma$) by

\[
\begin{align*}
  a \setminus \emptyset &= \emptyset, \\
  a \setminus \varepsilon &= \emptyset, \\
  a \setminus a &= \varepsilon, \\
  a \setminus b &= \emptyset \text{ for } a \neq b, \\
  a \setminus (E + F) &= a \setminus E + a \setminus F, \\
  a \setminus (EF) &= \begin{cases} 
  (a \setminus E)F + a \setminus F & \text{if } \varepsilon \in L(E), \\
  (a \setminus E)F & \text{otherwise},
  \end{cases} \\
  a \setminus (E^*) &= (a \setminus E)E^*.
\end{align*}
\]
Derivative of a Regular Expression

Definition (Derivative of a regular expression by a string)
We define recursively $w \backslash E$ (derivative of the regular expression $E$ by $w \in \Sigma^*$) by

\[
\begin{align*}
\varepsilon \backslash E &= E, \\
ax \backslash E &= a \backslash (x \backslash E).
\end{align*}
\]
Derivative of a Regular Expression

Definition (Derivative of a regular expression by a string)

We define recursively \( w\backslash E \) (derivative of the regular expression \( E \) by \( w \in \Sigma^* \)) by

\[
\varepsilon\backslash E = E, \\
ax\backslash E = a\backslash (x\backslash E).
\]

Theorem (Brzozowski’s Theorem 4.2)

\( w \in L(E) \iff \varepsilon \in L(w\backslash E) \).
References
