Automata and Formal Languages - CM0081
A Language That Is Not Recursively Enumerable

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Recall

- A language $L$ is recursively enumerable iff exists a Turing machine $M$ such that $L = L(M)$.

A Language That Is Not Recursively Enumerable
Undecidability

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- A language $L$ is recursive iff exists a Turing machine $M$ such that
  i) $L = L(M)$ and
  ii) $M$ always halt (even if it does not accept).

Definition

A language $L$ is undecidable iff $L$ is not recursive.
Undecidability

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- A function is recursive if only if it is Turing-machine computable (see, for example, [Boolos, Burges and Jeffrey 2007], [Hermes 1969] or [Kleene 1952]).
- Recursive problem: “it is sufficiently simple that I can write a recursive function to solve it, and the function always finishes.” [Hopcroft, Motwani and Ullman 2007, p. 385]
Codification of Turing Machines

Convention
The Turing machine $M$ is of the form:

$$M = (\{q_1, \ldots, q_n\}, \{0, 1\}, \{X_1, X_2, X_3, \ldots, X_m\}, \delta, q_1, B, \{q_2\}),$$

where $X_1 = 0$, $X_2 = 1$ and $X_3 = B$. Moreover, $D_1 = L$ and $D_2 = R$. 
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Codification of an instruction

The instruction $\delta(q_i, X_j) = (q_k, X_l, D_m)$ is codified by

$$0^i10^j10^k10^l10^m.$$
Codification of a Turing machine

Let $C_1, C_2, \ldots, C_p$ be the codifications of the instructions of a Turing machine $M$. The codification of $M$ is

$$\overrightarrow{M} = C_1 11C_2 11 \ldots 11C_p.$$
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$$

Remark

Note that there are other possible codes for $M$. 
A enumeration for the binary strings
We ordered the binary strings by [length-]lexicographical order (strings are ordered by length, and strings of equal length are ordered lexicographically).
If $w$ is a binary string, we call $w$ the $i$-th string where $1w$ is the binary integer $i$. We refer to the $i$-th string as $w_i$. 

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\( i \)-th Turing machine

Given a Turing machine \( M \) with code \( w_i \), we can now associate a natural number to it: \( M \) is the \( i \)-th Turing machine, referred to as \( M_i \).
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Convention
If $w_i$ is not a valid Turing machine code, we shall take $M_i$ to be the Turing machine with one state and no transitions. Hence $L(M_i) = \emptyset$ if $w_i$ is not a valid Turing machine code. 
The Diagonalization Language

The diagonalization language
A language that is not recursively enumerable.

\[ L_d = \{ w_i \mid w_i \notin L(M_i) \}. \]
# The Diagonalization Language

## The diagonalization language

A language that is not recursively enumerable.

\[
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\]

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<thead>
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<th></th>
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<th>2</th>
<th>3</th>
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| ... | ... | ... | ... | ... | ...

\[
a_{ij} = \begin{cases} 
1, & \text{if } w_j \in L(M_i); \\
0, & \text{if } w_j \notin L(M_i). 
\end{cases}
\]

Vector for the language \(L(M_i): i\)-th row

\(L_d\): Complement of the diagonal

Is it possible that \(L_d\) be in a row?
The Diagonalization Language

Theorem (9.2)

$L_d$ is not recursively enumerable.

Proof

Whiteboard.
References


